Calculus

Limits:

$$\lim_{b \to a} \left(\frac{f(b) - f(a)}{b - a} \right)$$

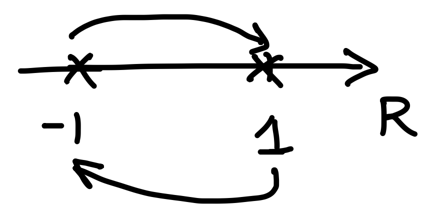
Derivatives → optimization problems.

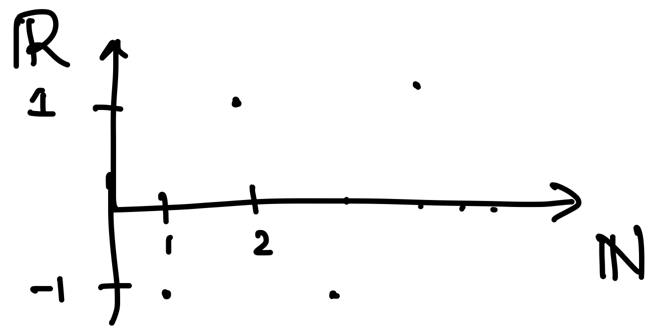
Integrals → expectation

Sequences

Examples: (a) $(a_n)_{n\in\mathbb{N}} = \left((-1)^n\right)_{n\in\mathbb{N}}$

$$= (-1, 1, -1, 1, \ldots)$$





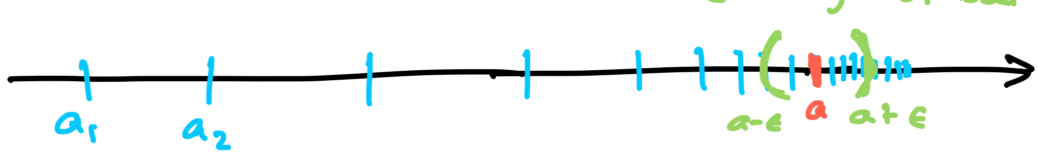
(b) $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$

$$\lim_{n\to\infty} a_n = 0$$

(c) $(a_n)_{n\in\mathbb{N}} = \left(2^n\right)_{n\in\mathbb{N}} = (2, 4, 8, 16, \ldots)$

Def: A sequence $(a_n)_{n\in\mathbb{N}}$ is called <u>convergent</u> <u>to $a\in\mathbb{R}$</u> if $\forall \varepsilon > 0\ \exists N\in\mathbb{N}\ \forall n \geq N : |a_n - a| < \varepsilon$

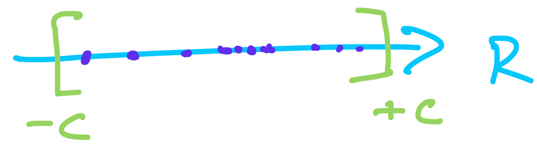
ε-neighborhood around 'a'



If there is no such $a\in\mathbb{R}$, then sequence diverges.

Def: A sequence $(a_n)_{n\in\mathbb{N}}$ is called <u>bounded</u> if $\exists C \in \mathbb{R}\ \forall n \in \mathbb{N}: \ |a_n| \leq C$

otherwise, the sequence is unbounded.

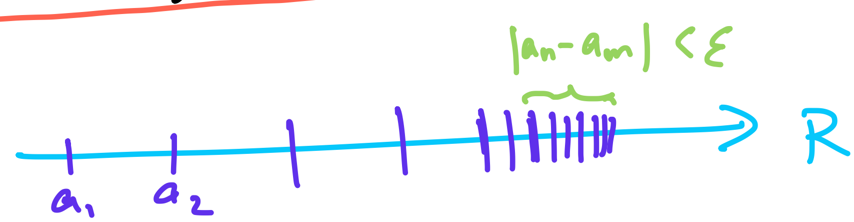


Facts: $(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ bounded.

$(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ There is only one limit $a \in \mathbb{R}$

$\lim\limits_{n\to\infty} a_n = a$

Def: If $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N:$ $|a_n - a_m| < \varepsilon$. then $(a)_{n\in\mathbb{N}}$ is called a <u>Cauchy Sequence.</u>



Fact: For a sequence of <u>real</u> numbers:

$$\text{Cauchy Sequence} \iff \text{convergent sequence}$$

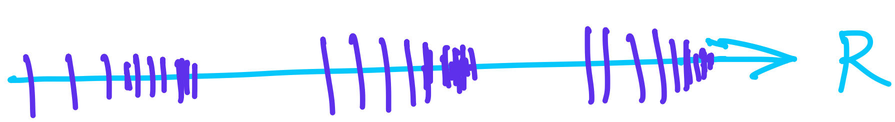
Prop: If $(a_n)_{n \in \mathbb{N}}$ is montonically decreasing ($a_{n+1} \leq a_n \ \forall n$) and bounded from below (the set $\{a_n\}_{n \in \mathbb{N}}$ has a lower bound), then $(a_n)_{n \in \mathbb{N}}$ is convergent.

Example Subsequence: $(a_n)_{n \in \mathbb{N}} = (-1)^n$

subsequence: $(a_n)_{n \in \mathbb{N}} = (a_{2k})_{k \in \mathbb{N}} = (1, 1, \dots, 1) \to 1$

subsequence: $(a_n)_{n \in \mathbb{N}} = (a_{2k+1})_{k \in \mathbb{N}} = (-1, -1, \dots) \to -1$

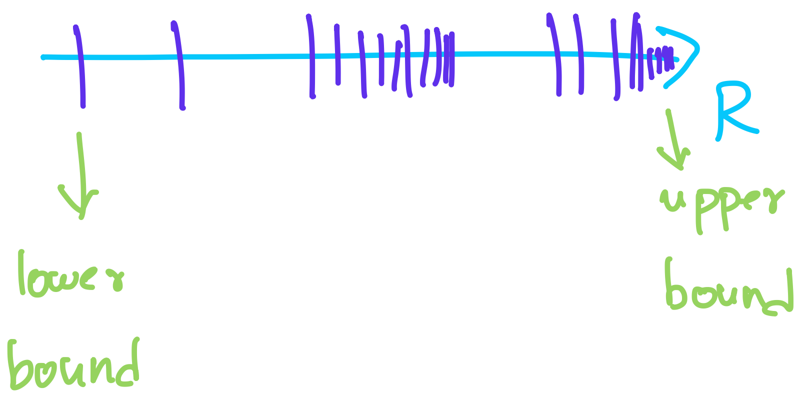
Def: $a \in \mathbb{R}$ is called an accumulation value of $(a_n)_{n \in \mathbb{N}}$ if there is a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} a_{n_k} = a$.



(cluster point, accumulation point, limit point, partial limit ...)

Theorem: Bolzano-Weierstrass theorem.

$(a_n)_{n\in\mathbb{N}}$ bounded $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value (has a convergent subsequence.)



Observations:

- a sequence can have many acc. points (or no accumulation point)
- even if the sequence has just one acc. point, it is not necessarily a Cauchy sequence.
- If $(a_n)_{n\in\mathbb{N}}$ converges to a, then a is the only acc. point. & the sequence is Cauchy sequence.

Example: $(a_n)_{n\in\mathbb{N}} = \frac{1}{n}$ on $(0,1]$

$(a_n)_{n\in\mathbb{N}}$ is Cauchy, but does not converge on $(0,1]$. It does converge to 0 on $[0,1]$.

Max, Sup, Min, Inf

Assume we are on $\mathbb{R}$ (or more generally, on a space that has a total ordering).
Let $U \subset \mathbb{R}$ be a subset.

- $x \in \mathbb{R}$ is called a maximum element of U if $x \in U$ and $\forall u \in U: \ u \leq x$.

  1 is max $[0,1]$
  $(0,1)$ has no max

- x is called an upper bound of U if $\forall u \in U: \ u \leq x$

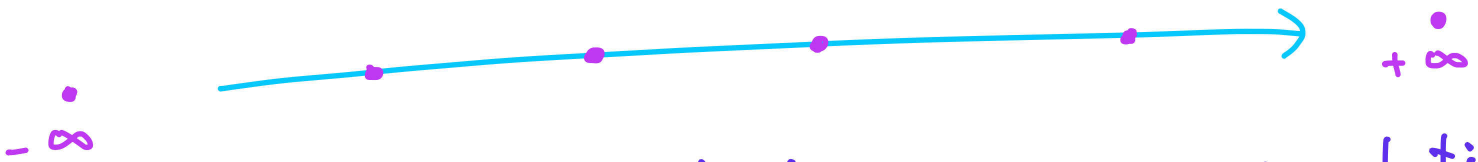
  5 is an upper bound of $(0,1)$ or $[0,1]$

- x is called a supremum of U if it is the smallest upper bound.

  1 is the sup. of $(0,1)$

Analogously define minimum, lower bound and infimum.

A given sequence $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values:



$+\infty, -\infty$ are called improper accumulation points.

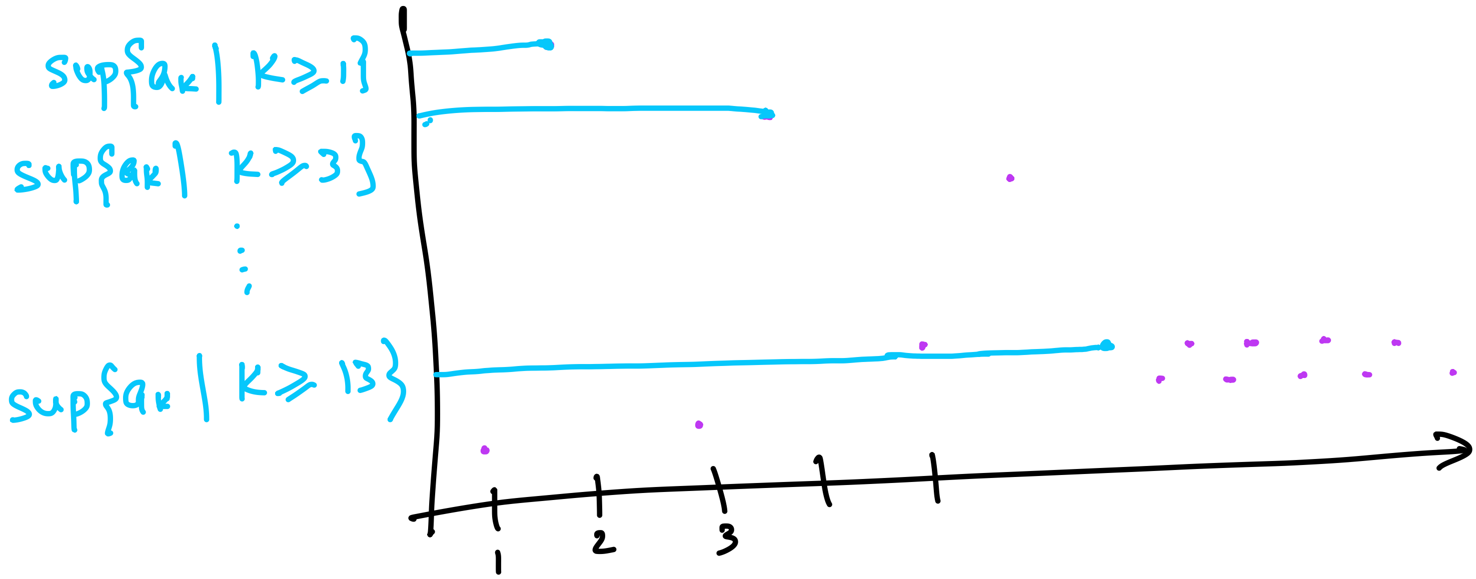
Def: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. An element $a \in \mathbb{R} \cup \{-\infty, +\infty\}$ is called:

- limit superior of $(a_n)_{n\in\mathbb{N}}$ if a is the largest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

  write $a = \limsup\limits_{n\to\infty} a_n$

- limit inferior of $(a_n)_{n\in\mathbb{N}}$ if a is the smallest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

  write $a = \liminf\limits_{n\to\infty} a_n$



<u>Fact</u>:

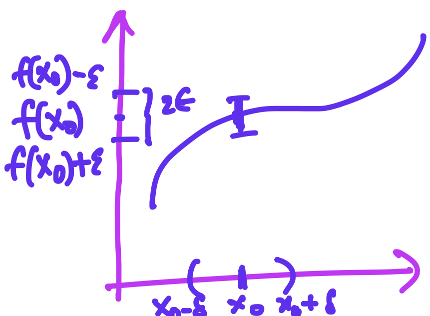
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$

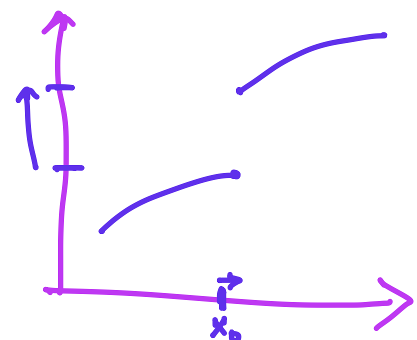
Continuity

Def: A function $f: X \to Y$ between two metric spaces (X, d), (Y, d) is called continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X: \ d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$



continuous



not continuous.

Alternative Def: $f: X \to Y$ is called continuous at x_0 if for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ we have:

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$$

A function $f: X \to Y$ is called continuous if it is continuous for every $x_0 \in X$:

$$\forall x_0 \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X: d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

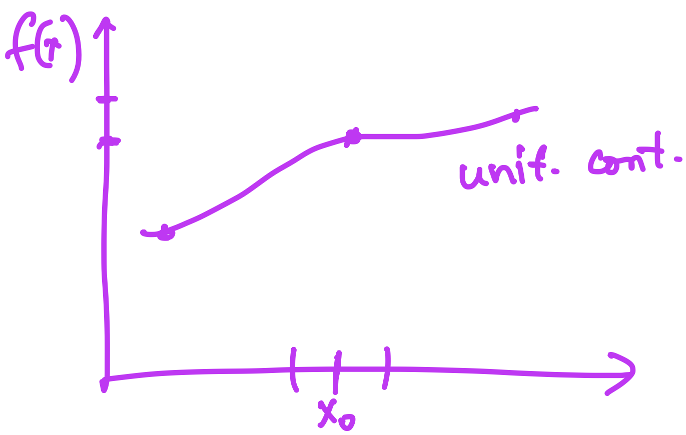
A function $f: X \to Y$ is called Lipschitz continuous with Lipschitz constant L if

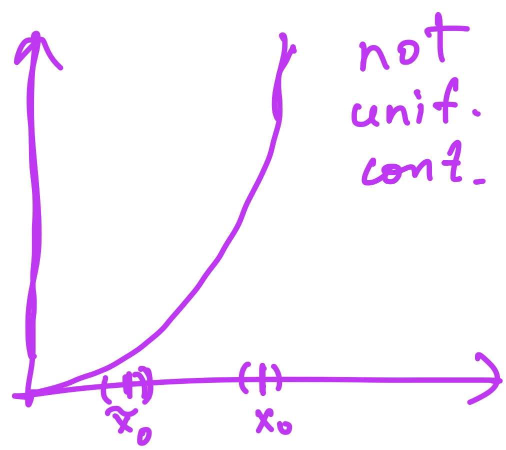
$$\forall x, y \in X: \quad d(f(x), f(y)) \leq L \cdot d(x, y)$$

Intuition: "bounded derivative"

A function $f: X \to Y$ is called uniformly continuous if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \underline{\forall x_0 \in X} \ \forall x \in X: \ d(x, x_0) < \delta$$
$$\Rightarrow \ d(f(x), f(x_0)) < \varepsilon$$





Given ε, I can choose δ that works for all x_0

Intuition: bounded derivative

Cannot choose δ to be the same for all x_0

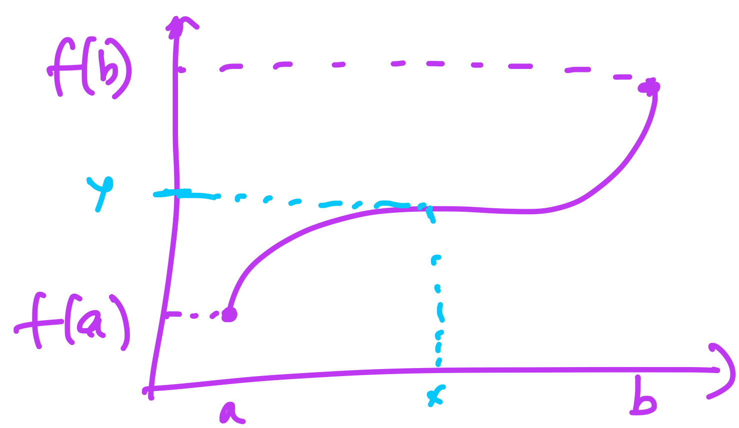
Intuition: unbounded derivative.

Important theorems for Continuous Funcs.

Intermediate value theorem:

If $f : [a,b] \to \mathbb{R}$ is continuous, then f attains all values between $f(a)$ & $f(b)$:

$$\forall y \in [f(a), f(b)] \ \exists x \in [a,b] : f(x) = y$$



Application: If you want to find x with $f(x) = 0$:

- find a with $f(a) < 0$, find b with $f(b) > 0$

then there must exist $x \in [a,b]$ with

$$f(x) = 0$$

Invertible Functions: $D \subset \mathbb{R}$, $f: D \to \mathbb{R}$ continuous, strictly monotone ($a < b \Rightarrow f(a) < f(b)$)

Then f is invertible and the inverse is continuous as well.

- Invertible follows from monotonicity

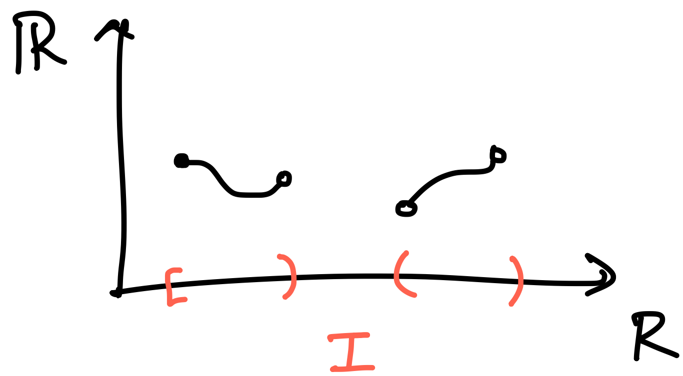
- continuity of the inverse follows directly from continuity of f.

A function f between two metric spaces (X, d), (Y, d) is continuous if and only if pre-images of open sets are open:

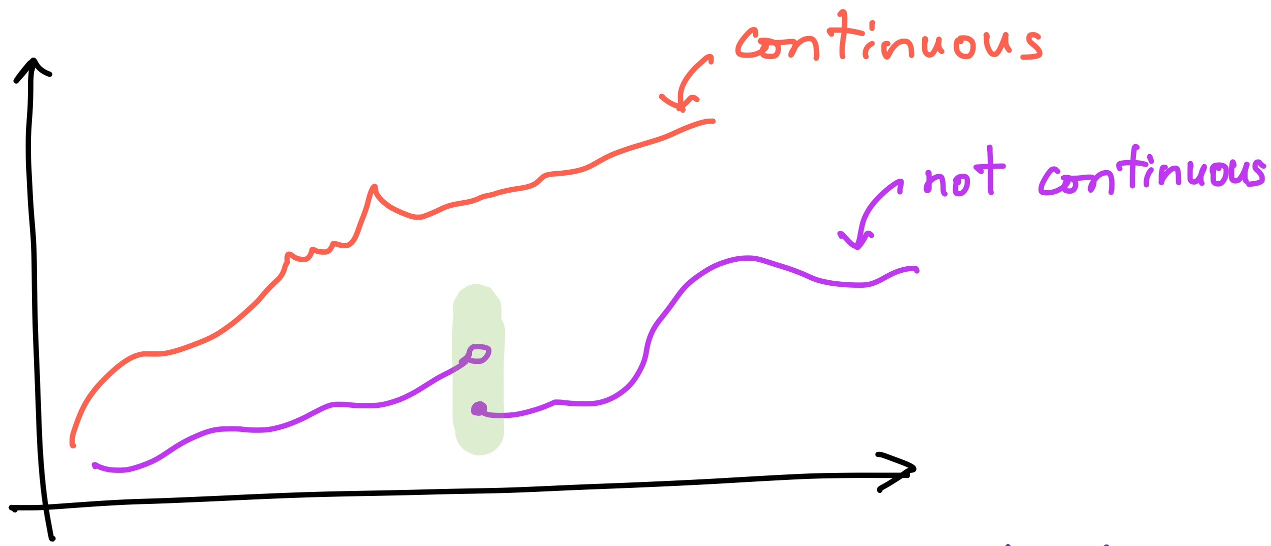
$$\underset{\text{in } Y}{B \subset Y \text{ open}} \Rightarrow f^{-1}(B) := \{x \in X \mid f(x) \in B\} \text{ open in } X.$$

Sequences of functions

Function: $f : I \to \mathbb{R}$ $\quad (I \subseteq \mathbb{R})$.

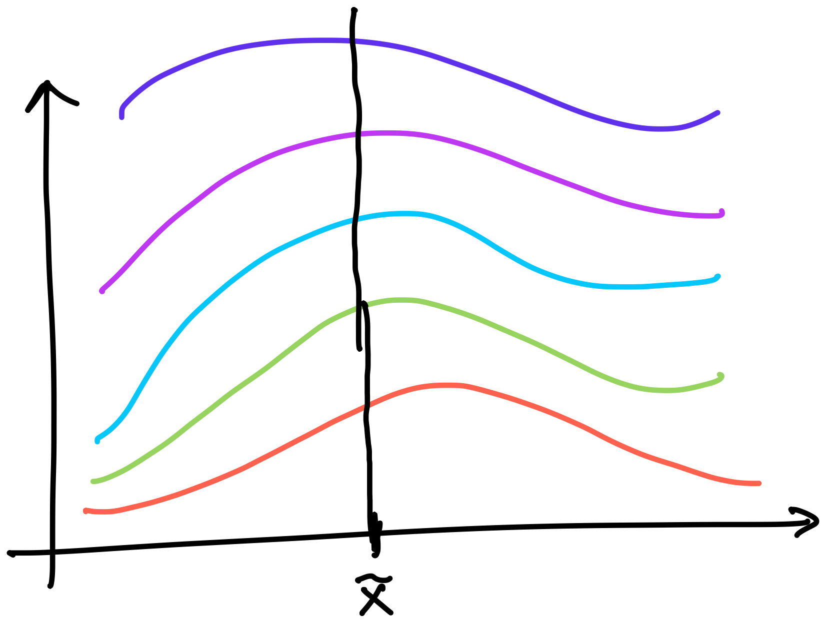


continuous functions: $f : \mathbb{R} \to \mathbb{R}$



Idea: small changes on x-axis → small changes on y-axis.

Sequences of functions



sequence:

$(f_1, f_2, f_3, f_4 \dots)$

with members

$f_1 : I \to \mathbb{R}$

$f_2 : I \to \mathbb{R}$

$\vdots$

For any fixed $\tilde{x} \in I$, we can get an ordinary sequence of real-numbers.

$$\left(f_1(\tilde{x}) \,,\, f_2(\tilde{x}) \,,\, f_3(\tilde{x}) \,,\, f_4(\tilde{x}) ,\, \dots \right)$$

Def: Consider functions: $f_n : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$

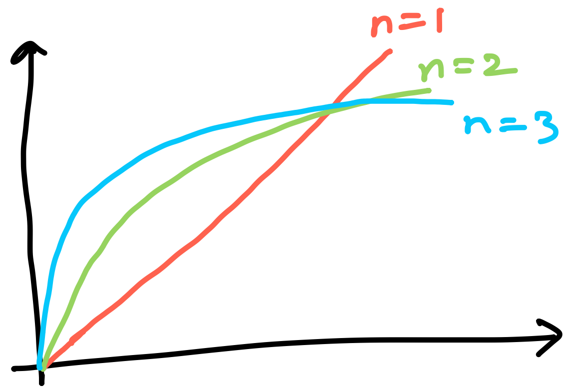
We say that the sequence $(f_n)_{n \in \mathbb{N}}$ converges pointwise to $f : I \to \mathbb{R}$ if

$$\forall x \in I : \quad f_n(x) \longrightarrow f(x)$$

$$y_n := f_n(x) \,, \quad y = f(x)$$

$$y_n \longrightarrow y$$

Example: $f_n, f : [0,1] \to \mathbb{R}$, $f_n(x) = x^{1/n}$



$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & \text{otherwise} \end{cases}$$

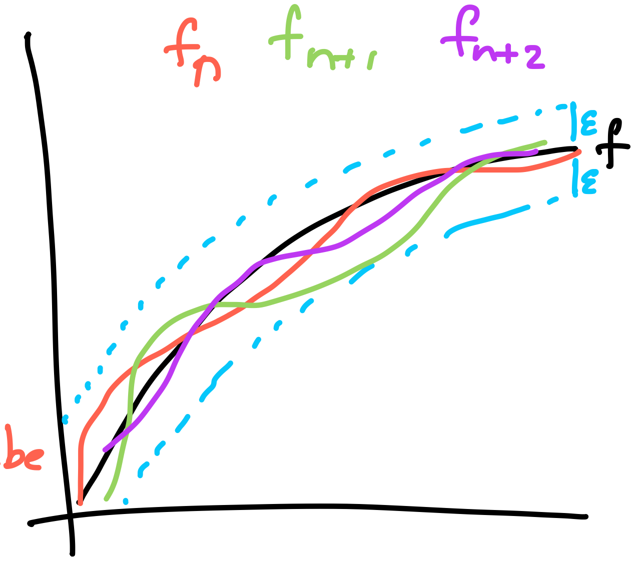
⚠ $f_n \to f$ pointwise, all f_n continuous, this does not imply that f is continuous

Def: $(f_n)_{n \in \mathbb{N}}$ converges to f uniformly if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ \forall x \in I : |f_n(x) - f(x)| < \varepsilon$$

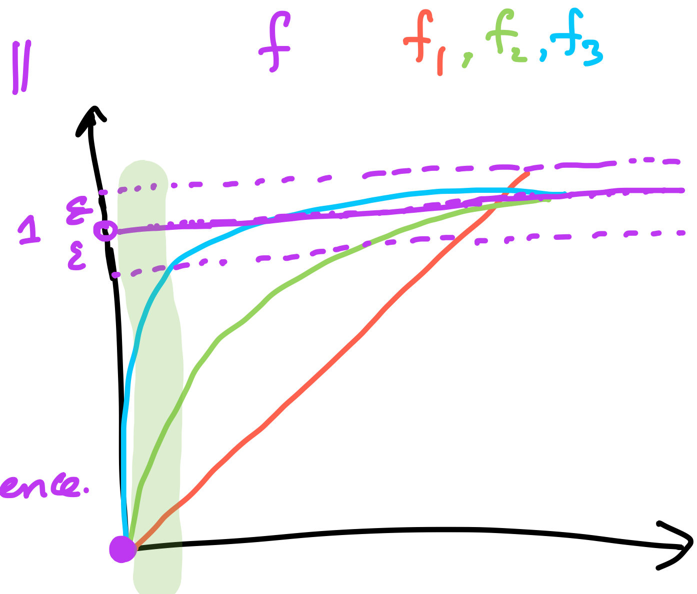
Intuition:

uniform convergence: given ε, there exist N such that all f_n $n > N$ are contained within ε-tube



Close to zero, there will always be points x close to zero such that the $f_n(x)$ are not yet in ε-tube.

$\Rightarrow$ Not uniformly convergence.



Alternative definition: $f_n \to f$ uniformly iff $\|f_n - f\|_\infty \to 0.$

Theorem: (uniform convergence preserves continuity)

$f_n, f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, all f_n are continuous, $f_n \to f$ uniformly. Then f is continuous.