

Calculus

Limits:

$$\lim_{b \rightarrow a} \left(\frac{f(b) - f(a)}{b - a} \right)$$

Derivatives

→ optimization problems.

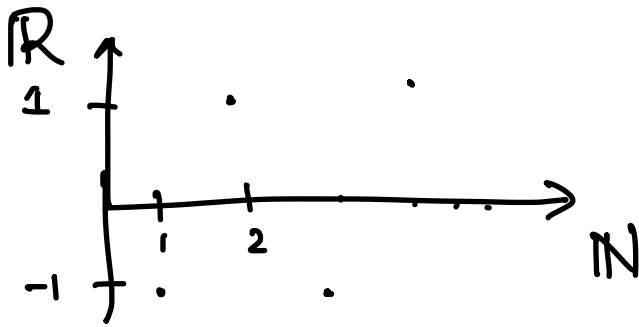
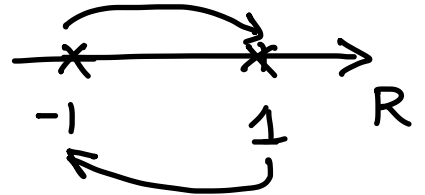
Integrals

→ expectation

Sequences

Examples: (a) $(a_n)_{n \in \mathbb{N}} = ((-1)^n)_{n \in \mathbb{N}}$

$$= (-1, 1, -1, 1, \dots)$$



(b) $(a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$

$$\lim_{n \rightarrow \infty} a_n = 0$$

(c) $(a_n)_{n \in \mathbb{N}} = (2^n)_{n \in \mathbb{N}} = (2, 4, 8, 16, \dots)$

Def: A sequence $(a_n)_{n \in \mathbb{N}}$ is called convergent

to $a \in \mathbb{R}$ if $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : |a_n - a| < \epsilon$

ϵ = neighborhood around 'a'

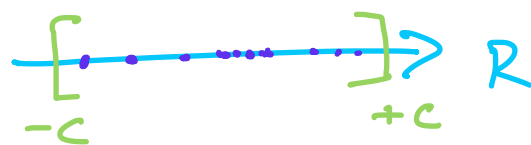


If there is no such $a \in \mathbb{R}$, then sequence diverges.

Def: A sequence $(a_n)_{n \in \mathbb{N}}$ is called bounded

if $\exists C \in \mathbb{R} \forall n \in \mathbb{N} : |a_n| \leq C$

otherwise, the sequence is unbounded.

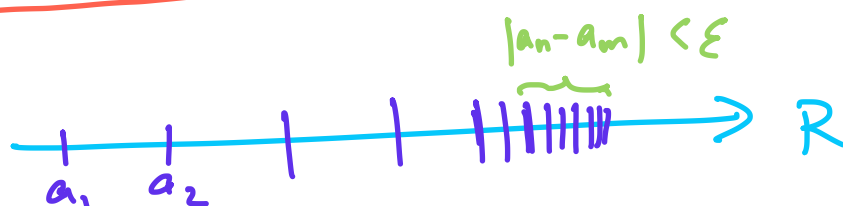


Facts: $(a_n)_{n \in \mathbb{N}}$ convergent $\Rightarrow (a_n)_{n \in \mathbb{N}}$ bounded.

$(a_n)_{n \in \mathbb{N}}$ convergent \Rightarrow There is only one limit
 $a \in \mathbb{R}$
 \downarrow
 $\lim_{n \rightarrow \infty} a_n = a$

Def: If $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N :$
 $|a_n - a_m| < \varepsilon$. then $(a_n)_{n \in \mathbb{N}}$ is called

a Cauchy Sequence.



Fact: For a sequence of real numbers:

Cauchy Sequence \Leftrightarrow convergent sequence

Prop: If $(a_n)_{n \in \mathbb{N}}$ is monotonically decreasing
 $(a_{n+1} \leq a_n \forall n)$ and bounded from below
 (the set $\{a_n\}_{n \in \mathbb{N}}$ has a lower bound), then
 $(a_n)_{n \in \mathbb{N}}$ is convergent.

Example Subsequence: $(a_n)_{n \in \mathbb{N}} = (-1)^n$
 subsequence: $(a_n)_{n \in \mathbb{N}} = (a_{2k})_{k \in \mathbb{N}} = (1, 1, \dots, 1) \rightarrow 1$

subsequence: $(a_n)_{n \in \mathbb{N}} = (a_{2k+1})_{k \in \mathbb{N}} = (-1, -1, \dots, -1) \rightarrow -1$

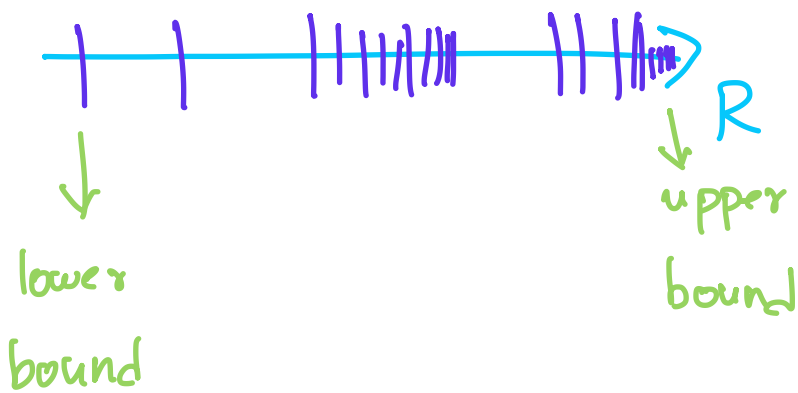
Def: $a \in \mathbb{R}$ is called an accumulation
value of $(a_n)_{n \in \mathbb{N}}$ if there is a subsequence
 $(a_{n_k})_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} a_{n_k} = a$.



(cluster point
 accumulation point,
 limit point,
 partial limit
 ...)

Theorem: Bolzano-Weierstrass theorem.

$(a_n)_{n \in \mathbb{N}}$ bounded $\Rightarrow (a_n)_{n \in \mathbb{N}}$ has an accumulation value



(has a convergent subsequence.)

Observations:

- a sequence can have many acc. points (or no accumulation point)
- even if the sequence has just one acc. point, it is not necessarily a Cauchy sequence.
- If $(a_n)_{n \in \mathbb{N}}$ converges to a , then a is the only acc. point. & the sequence is Cauchy sequence.

Example: $(a_n)_{n \in \mathbb{N}} = \frac{1}{n}$ on $(0, 1]$

$(a_n)_{n \in \mathbb{N}}$ is Cauchy, but does not converge on $(0, 1]$. It does converge to 0 on $[0, 1]$.

Max, Sup, Min, Inf

Assume we are on \mathbb{R} (or more generally, on a space that has a total ordering).

Let $U \subset \mathbb{R}$ be a subset.

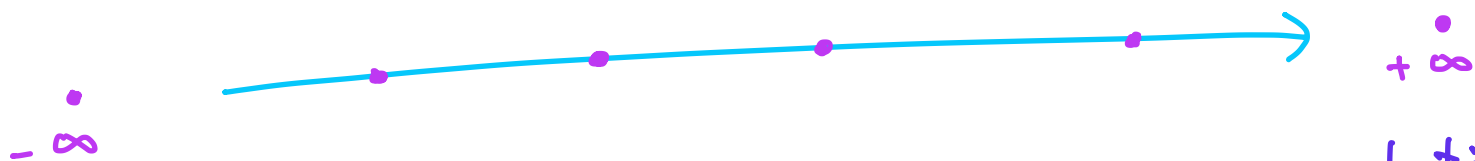
• $x \in \mathbb{R}$ is called a maximum element of U if $x \in U$ and $\forall u \in U: u \leq x$. 1 is max $[0, 1]$
 $(0, 1)$ has no max

• x is called an upper bound of U if $\forall u \in U: u \leq x$. 5 is an upper bound of $(0, 1)$ or $[0, 1]$

• x is called a supremum of U if it is the smallest upper bound. 1 is the sup. of $(0, 1)$

Analogously define minimum, lower bound and infimum.

A given sequence $(a_n)_{n \in \mathbb{N}}$ could have many accumulation values:



$+\infty, -\infty$ are called improper accumulation points.

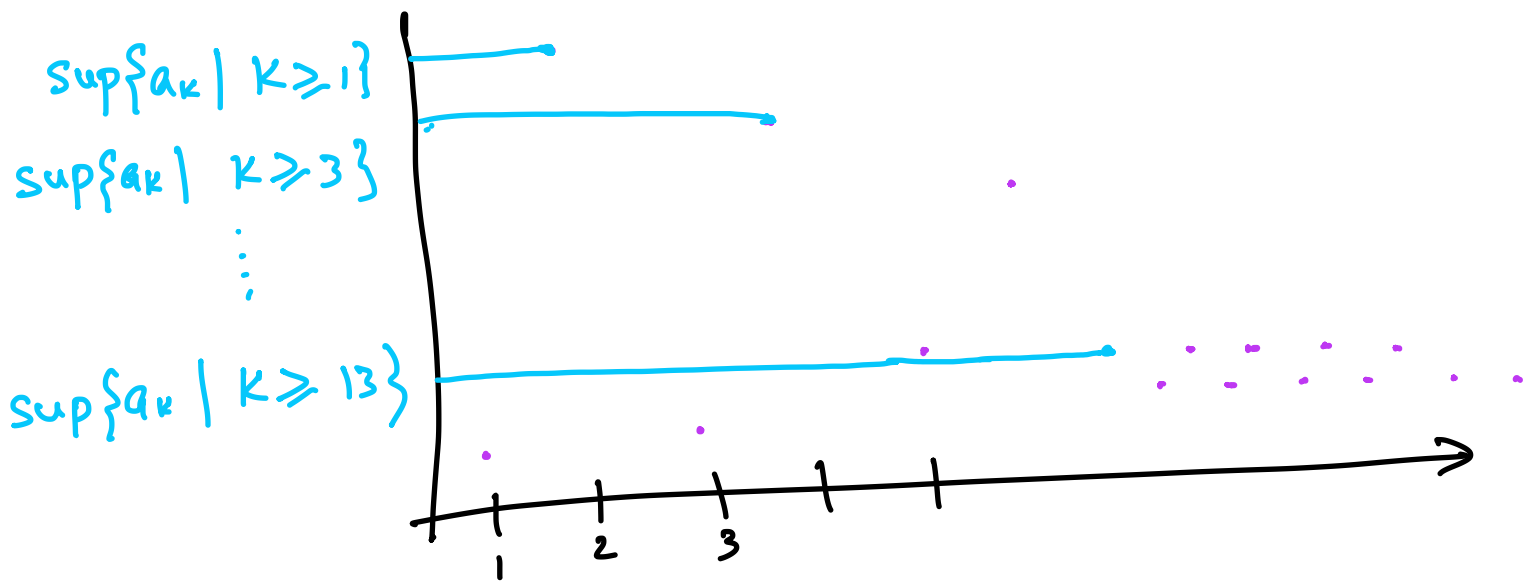
Def: Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. An element $a \in \mathbb{R} \cup \{-\infty, +\infty\}$ is called:

- limit superior of $(a_n)_{n \in \mathbb{N}}$ if a is the largest (improper) accumulation value of $(a_n)_{n \in \mathbb{N}}$

$$\text{write } a = \limsup_{n \rightarrow \infty} a_n$$

- limit inferior of $(a_n)_{n \in \mathbb{N}}$ if a is the smallest (improper) accumulation value of $(a_n)_{n \in \mathbb{N}}$

$$\text{write } a = \liminf_{n \rightarrow \infty} a_n$$



Fact: $\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup\{a_k \mid k \geq n\}$

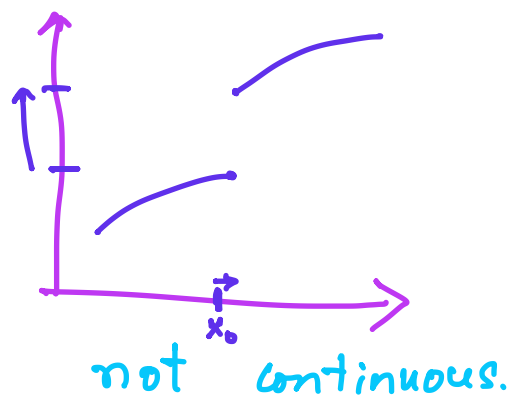
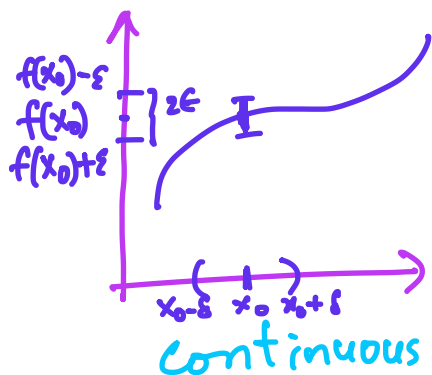
$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_k \mid k \geq n\}$

Continuity

Def: A function $f: X \rightarrow Y$ between two metric spaces (X, d) , (Y, d) is called continuous at

$x_0 \in X$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X: d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \underline{\varepsilon}$$



Alternative Def: $f: X \rightarrow Y$ is called continuous

at x_0 if for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$

we have: $x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow f(x_0)$

A function $f: X \rightarrow Y$ is called continuous if it is continuous for every $x_0 \in X$:

$$\forall x_0 \in X \quad \forall \varepsilon > 0 \exists \delta > 0 \forall x \in X: d(x, x_0) < \delta \\ \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

A function $f: X \rightarrow Y$ is called Lipschitz continuous with Lipschitz constant L if

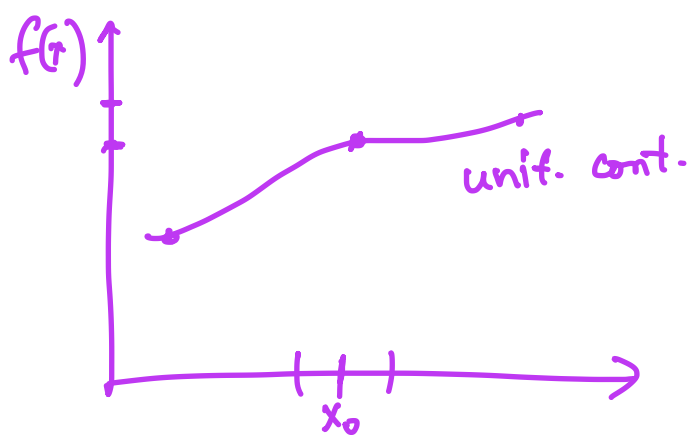
$$\forall x, y \in X: d(f(x), f(y)) \leq L \cdot d(x, y)$$

Intuition: "bounded derivative"

A function $f: X \rightarrow Y$ is called uniformly continuous if

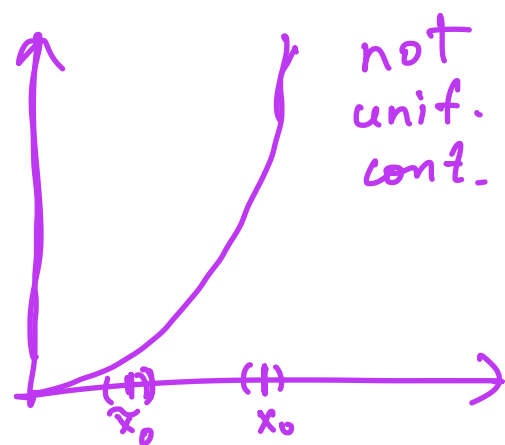
$$\forall \varepsilon > 0 \exists \delta > 0 \forall x_0 \in X \forall x \in X: d(x, x_0) < \delta$$

$$\Rightarrow d(f(x), f(x_0)) < \varepsilon$$



Given ε , I can choose δ that works for all x_0

Intuition: bounded derivative



Cannot choose δ to be the same for all x_0

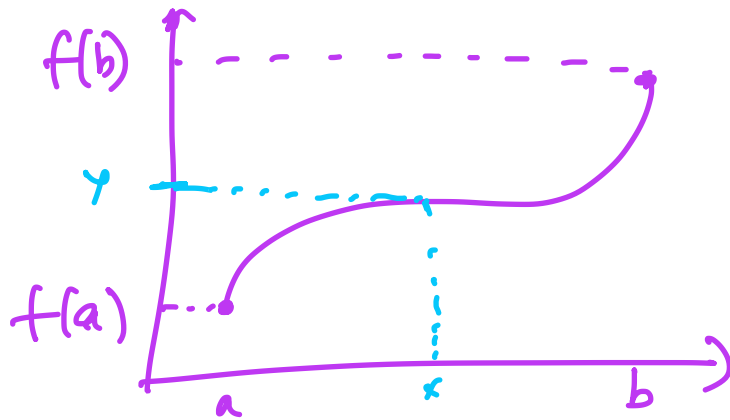
Intuition: unbounded derivative.

Important theorems for Continuous Funcs.

Intermediate value theorem:

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f attains all values between $f(a)$ & $f(b)$:

$$\forall y \in [f(a), f(b)] \exists x \in [a, b]: f(x) = y$$



Application: If you want to find x with $f(x) = 0$:
• find a with $f(a) < 0$,

there must exist $x \in [a, b]$ with $f(x) = 0$