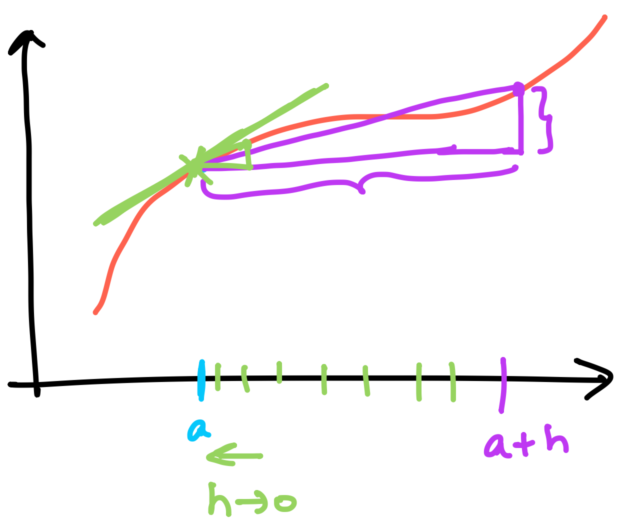
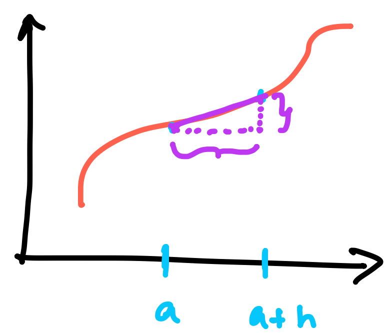
Derivatives (1-dim case)

Def: $U \subset \mathbb{R}$ an interval, $f: U \to \mathbb{R}$. The function is called differentiable at $a \in U$

if $$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ exists.

We often write $f' = \frac{df}{dx}$





$a_n : a + h_n$, $h_n \to 0$

$h_n > 0$

" derivative is slope of function at a "

" slope of linear approximation of function at a "

$$f(x) = f(a) + (x-a)\, \underline{b}$$

slope

derivative at 'a'

The function is called differentiable if it is differentiable for all $a \in U$. It is continuously differentiable if it is differentiable and the function $f' : U \to \mathbb{R}$, $a \mapsto f'(a)$ is continuous.

Higher derivatives: We can repeat the process of taking derivatives:

$$f' = \frac{df}{dx}, \quad f'' = \frac{df'}{dx}$$

Notation: $f^{(n)}$ denotes the n-th derivative (if exists).

Important Theorems

Theorem: (Differentiable $\overset{\cancel{\Leftarrow}}{\Rightarrow}$ continuous)

Let f be differentiable at a. Then there exists a constant C_a such that on a small ball around a, we have $|f(x) - f(a)| \leq C_a |x - a|$

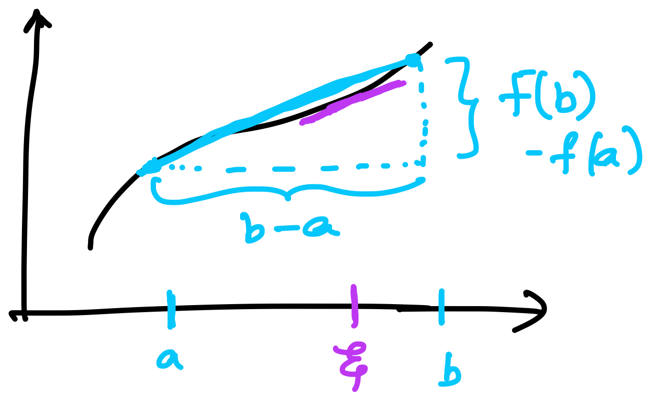
In particular, f is continuous at a.

Theorem: (intermediate value theorem for derivatives)

$f \in \mathcal{C}^1([a,b])$ (i.e. functions on $[a,b]$ that are once continuously differentiable)

Then there exists $\xi \in [a,b]$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi).$$



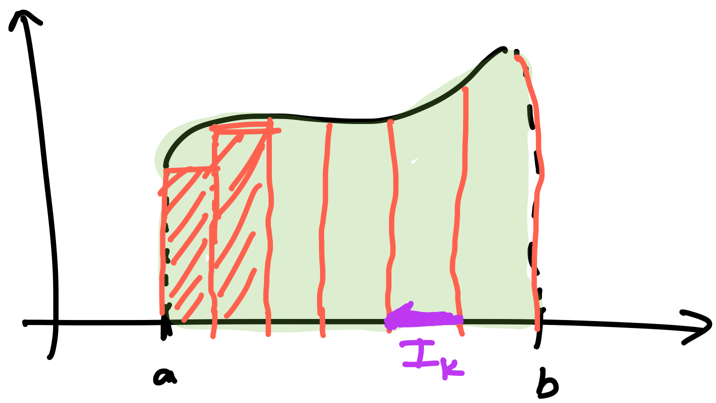
Theorem: (exchanging limits and derivatives)

$f_n : [a,b] \to \mathbb{R}$, $f_n \in \mathcal{C}^1([a,b])$. If the limit $f(x) := \lim_{n\to\infty} f_n(x)$ exists $\forall x \in [a,b]$ and the derivatives f' converge uniformly, then f is continuously differentiable and we have.

$$f'(x) = \left(\lim_{n\to\infty} f_n\right)'(x)$$ → first take limit of f_n, we obtain f, then compute derivative

$$= \left(\lim (f_n')\right)(x)$$ → first compute f_n', then take limit

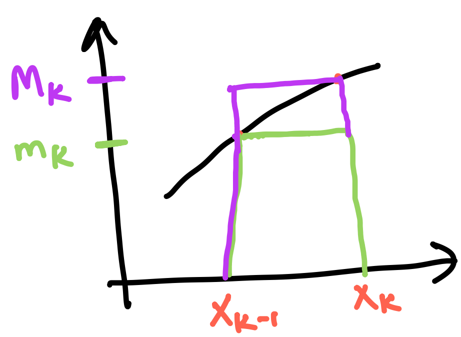
Riemann Integral



Consider a function $f: [a,b] \to \mathbb{R}$, assume that f is bounded ($\exists\, \ell, u \in \mathbb{R}\ \forall x \in [a,b]: \ell \leq f(x) \leq u$)

Consider $x_0, x_1, \dots, x_n$ with $a = x_0 < x_1 < x_2 \dots < x_n = b$

These points introduce a partition of $[a,b]$ into n intervals. $\quad I_k := [x_{k-1}, x_k]$



Define $m_k := \inf(f(I_k))$

$M_k := \sup(f(I_k))$

(exists since f is bounded)

Define the lower sum

length of $I_k = x_k - x_{k-1}$

$$s(f, \{x_0, x_1, \dots x_n\}) = \sum_{k=1}^{n} |I_k| \cdot m_k$$

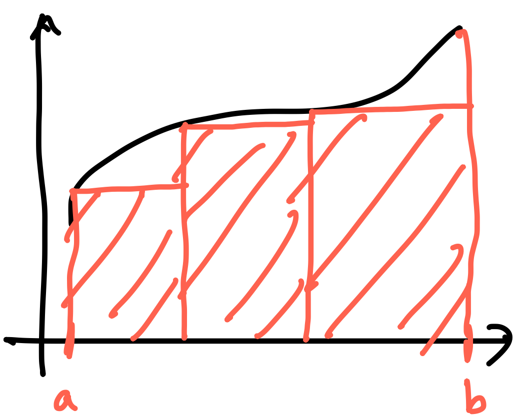
Define the upper sum,

$$S\left(f, \{x_0, x_1 \dots x_n\}\right) = \sum_{k=1}^{n} |I_k| \cdot M_k$$

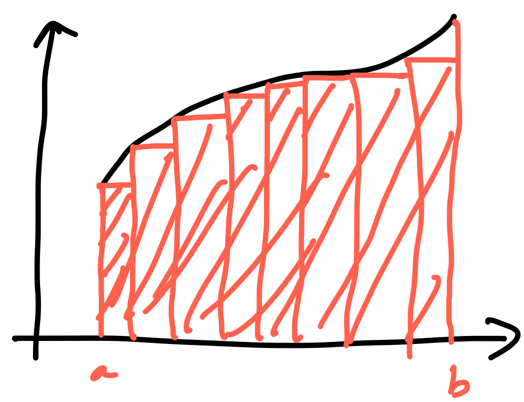
Now define

$$J_* := \sup_{\text{partitions}} \left(s(f, \text{partition}) \right)$$

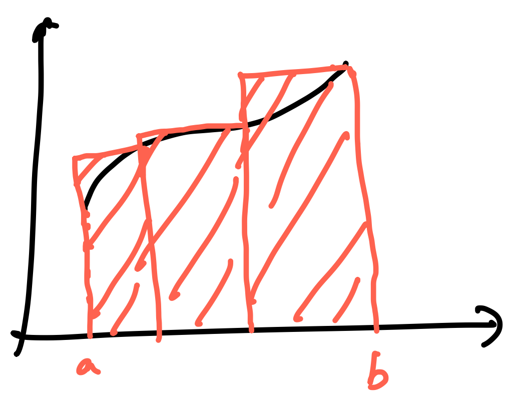
$$J^* := \inf_{\text{partitions}} \left(S(f, \text{partition}) \right)$$



coarse partition from below



finer partition from below.



partition from above

We call f Riemann-integrable if

$J_* = J^*$. Then we denote

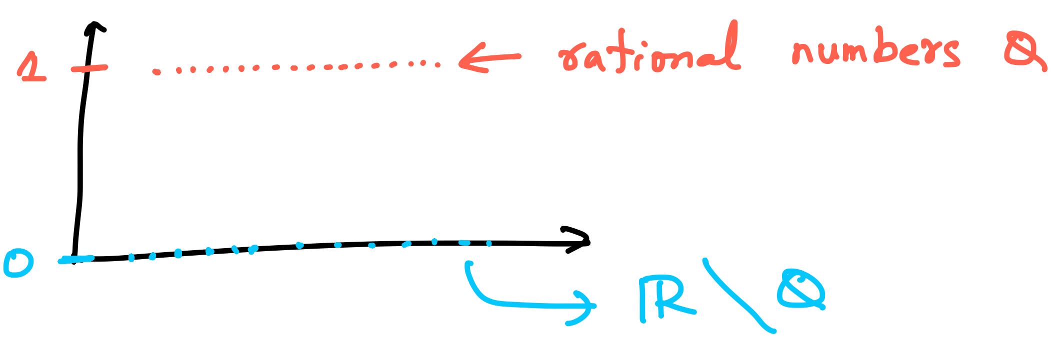
$$J_* = J^* := \int_a^b f(t)\,dt$$

Theorem:
- $f:[a,b]\to\mathbb{R}$ monotone $\Rightarrow$ integrable
  (i.e. $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$)
- $f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ integrable
  (true even if f is continuous everywhere except a finitely many points)

Shortcomings:
- Many functions are not integrable

Dirichlet Function: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{everywhere} \end{cases}$

For any interval
$I_k = [x_k, x_{k+1}]$
$M_k = 1$, $m_k = 0$



Then $\underset{\substack{\| \\ |b-a|\cdot 0}}{J_*} < \underset{\substack{\searrow \\ |b-a|\cdot 1}}{J^*}$

- One cannot prove theorems about exchanging "integral" with "lim": $\lim_{n\to\infty} \int f_n \, dt \stackrel{?}{=} \int \lim f_n \, dt$

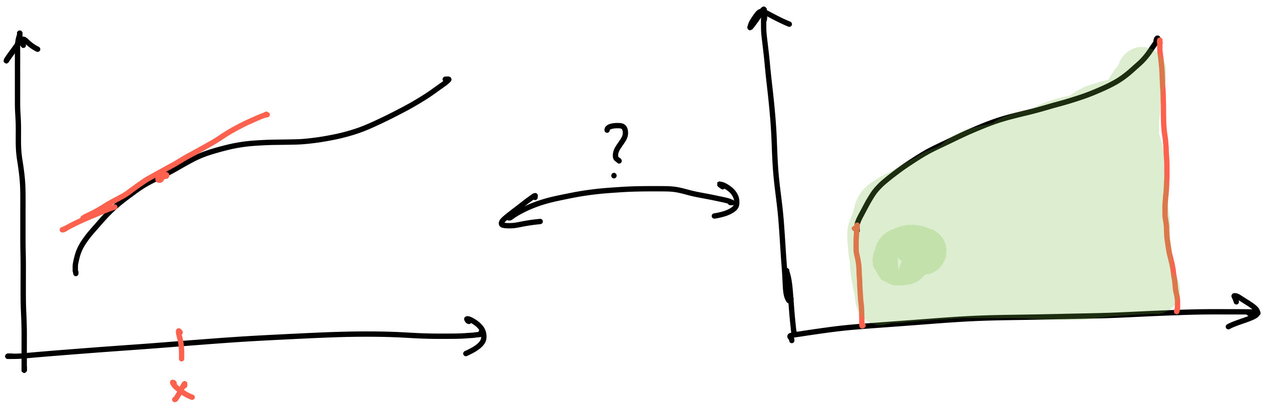
- Hard to extend to "other space"
(e.g. spaces with no notion of ordering, higher dimensional)

$\boxed{\text{Lebesgue Integration}}$

$\downarrow$

modern.

Fundamental Theorem of Calculus



Theorem I: $f:[a,b]\to\mathbb{R}$ (Riemann)-integrable and continuous at $\xi\in[a,b]$. Let $c\in[a,b]$. Then the function

$$F(x) := \int_c^x f(t)\,dt$$

is differentiable at ξ and $F'(\xi) = f(\xi)$.

cont. cont. differentiable.

If $f\in C([a,b])$, then $F\in C^1([a,b])$ and

$$F'(x) = f(x) \text{ for all } x\in[a,b].$$

Theorem II: $F:[a,b]\to\mathbb{R}$ continuously differentiable, then

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

Proof I: Need to prove that $F(x)$ is differentiable at ξ.

Consider $A(h) := \dfrac{F(\xi+h) - F(\xi)}{h}$

$$= \frac{1}{h}\left(\int_c^{\xi+h} f(t)\,dt - \int_c^{\xi} f(t)\,dt \right)$$

$$= \frac{1}{h} \int_{\xi}^{\xi+h} f(t)\,dt$$

want to prove: converges to $f(\xi)$ as $h \to 0$

Want to prove:

$$A(h) - f(\xi) \longrightarrow 0$$

$$f(\xi) = \frac{1}{h}\int_{\xi}^{\xi+h} \underbrace{f(\xi)}_{\text{const.}}\,dt$$

$$= \frac{1}{h} \cdot f(\xi) \cdot \int_{\xi}^{\xi+h} 1 \cdot dt$$

$$= \frac{1}{h} f(\xi) \cdot (\xi + h - \xi)$$

$$= f(\xi)$$

$$= \frac{1}{h}\int_{\xi}^{\xi+h} f(t)\,dt - f(\xi)$$

$$= \frac{1}{h}\int_{\xi}^{\xi+h} f(t)\,dt - \frac{1}{h}\int_{\xi}^{\xi+h} f(\xi)\,dt$$

$$= \frac{1}{h}\int_{\xi}^{\xi+h} \underbrace{\big(f(t) - f(\xi)\big)}\,dt$$

Intuition: small as $h \to 0$ since f is continuous at ξ.

Formally: given $\varepsilon > 0$ we can find $h > 0$ such that $f(t) - f(\xi) < \varepsilon \quad \forall\, t \in [\xi, \xi + h]$.

Then: $\displaystyle \frac{1}{h}\int_{\xi}^{\xi+h} (f(t) - f(\xi))\,dt \leq \frac{1}{h}\int_{\xi}^{\xi+h} |f(t) - f(\xi)|\,dt$

$$\leq \frac{1}{h}\int_{\xi}^{\xi+h} \varepsilon\, dt = \frac{1}{h}\cdot \varepsilon \int_{\xi}^{\xi+h} 1 \cdot dt = \frac{1}{h}\cdot \varepsilon \cdot h$$

$$= \varepsilon$$

$\Rightarrow \quad A(h) - f(\xi) \leq \varepsilon \to 0$ $\blacksquare$

Theorem I.

Proof II: Know that F' continuous. Then by Theorem I the function

$$G(x) := \int_a^x F'(t)\,dt \quad \text{is differentiable}$$

and (i) $G(a) = 0$ (by def. of G)

(ii) $G'(x) = F'(x)$ on $[a, b]$ (by Theorem I)

Consider $H(x) := F(x) - G(x)$

By (ii) we know that $H'(x) = F'(x) - G'(x) = 0 \;\forall x$.

Hence, H is a constant function.

We know that $H(a) = F(a) - \underbrace{G(a)}_{=0\ (i)} = F(a)$

(iii) $H(x) \equiv F(a)$ means : "constant"

Consider $x = b$.

$$F(a) \overset{(iii)}{=} H(b) \overset{def}{=} F(b) - G(b)$$

$$\overset{def}{=} F(b) - \int_a^b F'(t)\,dt$$

$$\Rightarrow \quad F(a) = F(b) - \int_a^b F'(t)\,dt$$

$$\Rightarrow \quad \int_a^b F'(t)\,dt = F(b) - F(a)$$

$\blacksquare$ Theorem II