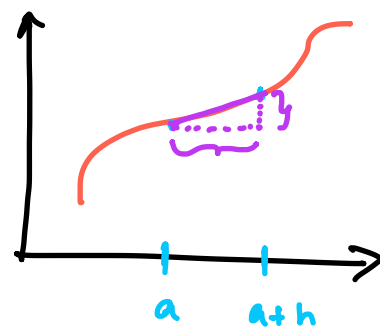
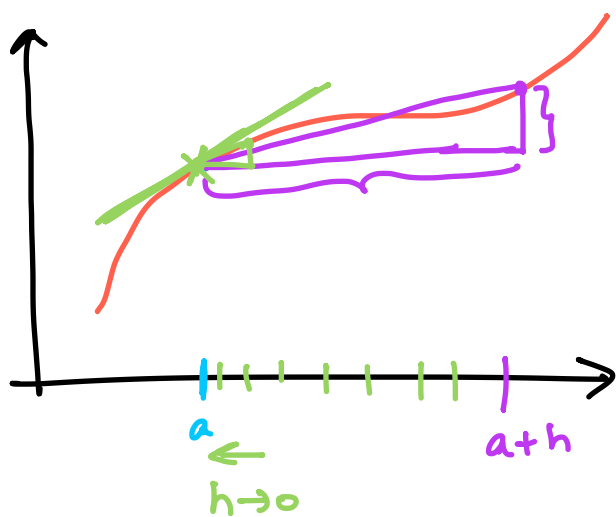


Derivatives (1-dim case)

Def: $U \subset \mathbb{R}$ an interval, $f: U \rightarrow \mathbb{R}$. The function is called differentiable at $a \in U$

if $f'(a) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists.

We often write $f' = \frac{df}{dx}$



$$a_n : a + h_n, \quad h_n \rightarrow 0 \\ h_n > 0$$

"derivative is slope of function at a "

"slope of linear approximation of function at a "

$$f(x) = f(a) + (x-a) \frac{b}{}$$

derivative at 'a' ← slope

The function is called differentiable if it is differentiable for all $a \in U$. It is continuously differentiable if it is differentiable and the function $f': U \rightarrow \mathbb{R}$, $a \mapsto f'(a)$ is continuous.

Higher derivatives: We can repeat the process of taking derivatives:

$$f' = \frac{df}{dx}, \quad f'' = \frac{df'}{dx}$$

Notation: $f^{(n)}$ denotes the n -th derivative (if exists).

Important Theorems

Theorem: (Differentiable ~~is~~ \Rightarrow continuous)

Let f be differentiable at a . Then there exists a constant C_a such that on a small ball around a ,

$$\text{we have } |f(x) - f(a)| \leq C_a |x - a|$$

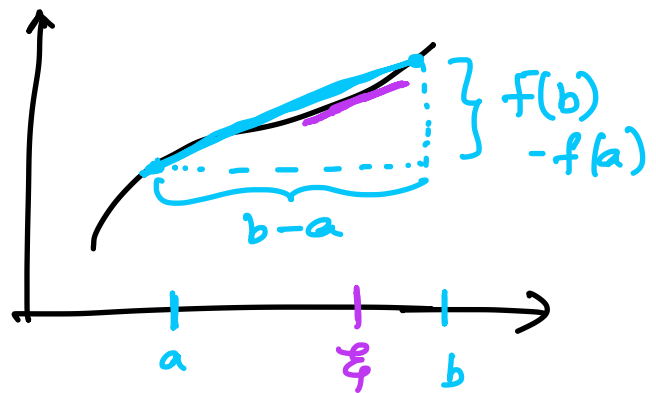
In particular, f is continuous at a .

Theorem: (intermediate value theorem for derivatives)

$f \in \mathcal{C}'([a, b])$ (i.e. functions on $[a, b]$ that are once continuously differentiable)

Then there exists $\xi \in [a, b]$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$



Theorem: (exchanging limits and derivatives)

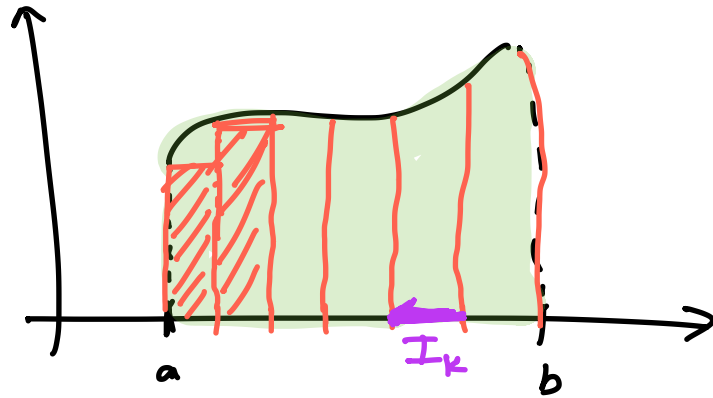
$f_n : [a, b] \rightarrow \mathbb{R}$, $f_n \in \mathcal{C}'([a, b])$. If the limit

$f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists $\forall x \in [a, b]$ and the

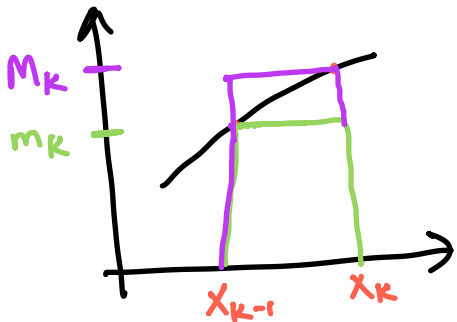
derivatives f'_n converge uniformly, then f is continuously differentiable and we have.

$$\begin{aligned} f'(x) &= \left(\lim_{n \rightarrow \infty} f_n \right)'(x) \rightarrow \text{first take limit of } f_n, \text{ we} \\ &= \left(\lim_{n \rightarrow \infty} (f'_n) \right)(x) \rightarrow \text{obtain } f, \text{ then compute derivative} \\ &\quad \rightarrow \text{first compute } f'_n, \text{ then take} \\ &\quad \quad \quad \text{limit} \end{aligned}$$

Riemann Integral



Consider a function $f: [a, b] \rightarrow \mathbb{R}$, assume that f is bounded ($\exists l, u \in \mathbb{R} \forall x \in [a, b]: l \leq f(x) \leq u$)
Consider x_0, x_1, \dots, x_n with $a = x_0 < x_1 < x_2 \dots < x_n = b$
These points introduce a partition of $[a, b]$ into n intervals. $I_k := [x_{k-1}, x_k]$



Define $m_k := \inf(f(I_k))$
 $M_k := \sup(f(I_k))$
(exists since f is bounded)

Define the lower sum

$$s(f, \{x_0, x_1, \dots, x_n\}) = \sum_{k=1}^n |I_k| \cdot m_k$$

length of $I_k = x_k - x_{k-1}$

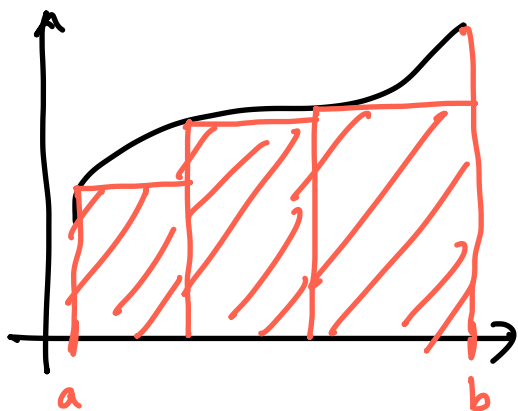
Define the upper sum,

$$S(f, \{x_0, x_1, \dots, x_n\}) = \sum_{k=1}^n |I_k| \cdot M_k$$

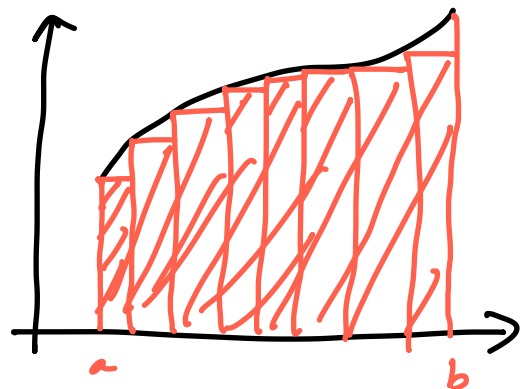
Now define

$$J_* := \sup_{\text{partitions}} (s(f, \text{partition}))$$

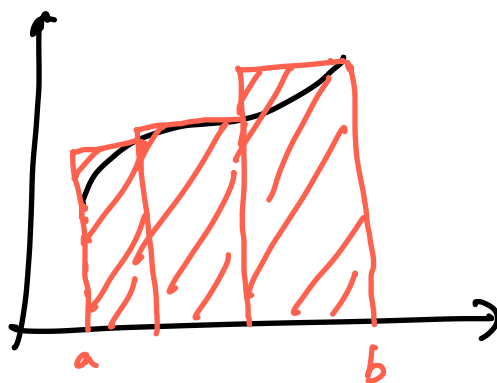
$$J^* := \inf_{\text{partitions}} (S(f, \text{partition}))$$



coarse partition
from below



finer partition from
below.



partition
from
above

We call f Riemann-integrable if

$J_* = J^*$. Then we denote

$$J_* = J^* := \int_a^b f(t) dt$$

Theorem: • $f: [a, b] \rightarrow \mathbb{R}$ monotone \Rightarrow integrable

(i.e. $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$)

• $f: [a, b] \rightarrow \mathbb{R}$ continuous \Rightarrow integrable

(true even if f is continuous everywhere
except a finitely many points)

Shortcomings:

• Many functions are not integrable

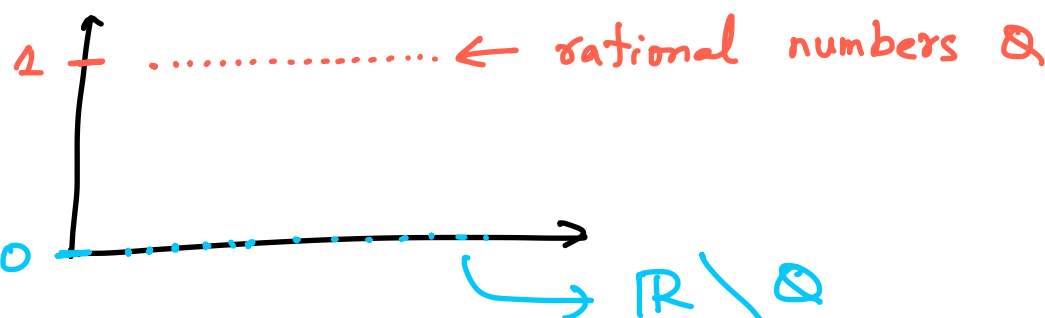
Dirichlet Function: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{everywhere} \end{cases}$

For any interval

$$I_k = [x_k, x_{k+1}]$$

$$M_k = 1, m_k = 0$$

Then $J_* < J^*$
" \parallel \downarrow
 $|b-a| \cdot 0$ \downarrow $|b-a| \cdot 1$

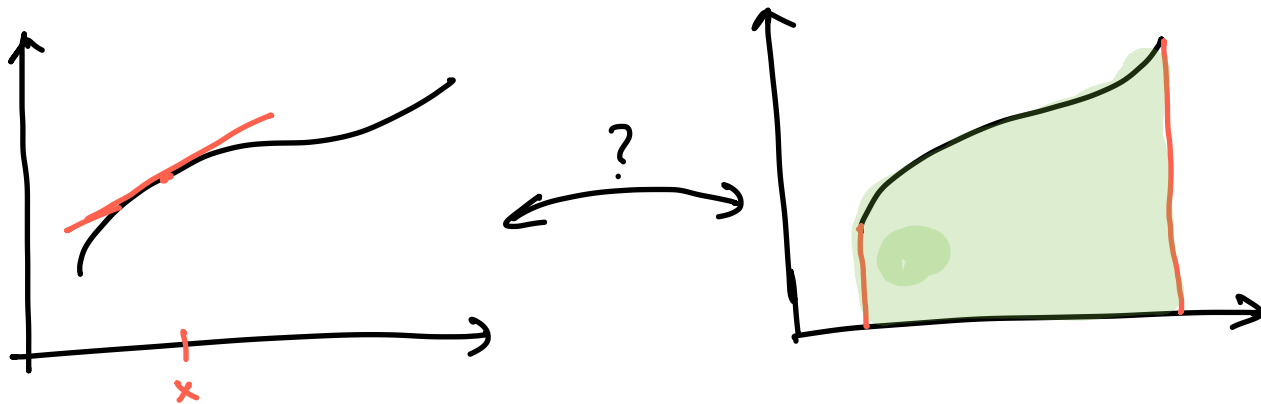


- One cannot prove theorems about exchanging "integral" with "lim": $\lim_{n \rightarrow \infty} \int f_n dt \stackrel{?}{=} \int \lim f_n dt$
- Hard to extend to "other space"
(e.g. spaces with no notion of ordering, higher dimensional)

Lebesgue Integration

↓
modern.

Fundamental Theorem of Calculus



Theorem I: $f: [a, b] \rightarrow \mathbb{R}$ (Riemann)-integrable and continuous at $\xi \in [a, b]$. Let $c \in [a, b]$.

Then the function

$$F(x) := \int_c^x f(t) dt$$

is differentiable at ξ and $F'(\xi) = f(\xi)$.

cont. \leftarrow If $f \in \mathcal{C}([a, b])$, then $F \in \mathcal{C}'([a, b])$ and \mathcal{C}' \rightarrow *cont. differentiable.*

$F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem II: $F: [a, b] \rightarrow \mathbb{R}$ continuously differentiable, then

$$\int_a^b F'(t) dt = F(b) - F(a)$$

Proof I: Need to prove that $F(x)$ is differentiable at ξ .

Consider $A(h) := \frac{F(\xi+h) - F(\xi)}{h}$

$$= \frac{1}{h} \left(\int_{\xi}^{\xi+h} f(t) dt - \int_{\xi}^{\xi} f(t) dt \right)$$

$$= \frac{1}{h} \int_{\xi}^{\xi+h} f(t) dt$$

want to prove:
→ converges to $f(\xi)$ as $h \rightarrow 0$

Want to prove:

$$A(h) - f(\xi) \rightarrow 0$$

$$\begin{aligned} &= \frac{1}{h} \int_{\xi}^{\xi+h} f(t) dt - f(\xi) \\ &= \frac{1}{h} \int_{\xi}^{\xi+h} f(t) dt - \frac{1}{h} \int_{\xi}^{\xi+h} f(\xi) dt \\ &= \frac{1}{h} \int_{\xi}^{\xi+h} \underbrace{(f(t) - f(\xi))}_{\text{red wavy line}} dt \end{aligned}$$

$$\begin{aligned} f(\xi) &= \frac{1}{h} \int_{\xi}^{\xi+h} \underbrace{f(\xi)}_{\text{const.}} dt \\ &= \frac{1}{h} \cdot f(\xi) \cdot \int_{\xi}^{\xi+h} 1 \cdot dt \\ &= \frac{1}{h} f(\xi) \cdot (\xi+h - \xi) \\ &= f(\xi) \end{aligned}$$

Intuition: small as $h \rightarrow 0$ since f is continuous at ξ .

Formally: given $\varepsilon > 0$ we can find $h > 0$
such that $f(t) - f(\xi) < \varepsilon \quad \forall t \in [\xi, \xi + h]$.

Hence, H is a constant function.