Power Series

Def: A series of the form $p(x) : \sum_{n=0}^{\infty} a_n x^n$ is called a power series.

↳ powers of 'x'

↳ infinite sum is called a series.

Theorem: (Radius of convergence)

For every power series $p(x) := \sum_{n=0}^{\infty} a_n x^n$ there exists a constant r, $0 \le r \le \infty$, called the radius of convergence such that

- The series converges (absolutely) for all x with $|x| < r$ (means that $\sum_{n=0}^{\infty} a_n |x|^n$ converges, the sequence of partial sums $P_N(x) := \sum_{n=0}^{N} a_n |x|^n$ converges "in the usual sense" as $N \to \infty$)

  ⚠ unclear what happens when $|x| = r$

- If $|x| < r$, the series even converges uniformly.

The radius of convergence only depends on $(a_n)_n$ and can be computed by various formulae.

- $r = \frac{1}{L}$ where $L = \limsup_{n\to\infty} \left(|a_n|\right)^{1/n}$
- $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

if exist

Examples: • $p(x) = \sum_{n=0}^{\infty} \underbrace{n^c}_{a_n} x^n$ for some constant 'c'

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n^c}{(n+1)^c} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^c \simeq 1$$

Case $c = -1$: $\sum \frac{1}{n} x^n$ has conv. radius $r = 1$

- For $x = +1$, the series diverges.

$$\sum \frac{1}{n} x^n = \sum \frac{1}{n} \cdot 1^n = \sum_{n=0}^{\infty} \frac{1}{n} \to \infty$$

- For $x = -1$ it converges.
- For $x > 1$ it diverges.

Case $c = 0$: $\sum n^c x^n = \sum x^n$ diverges for $|x| = r$. (both $x = +1$ and $x = -1$).

- Exponential Series :

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{has} \quad r = \infty$$

since $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1/n!}{1/(n+1)!} = n+1 \to \infty$

- $\sum_{n=0}^{\infty} n!\, x^n$ has $r = 0$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} \to 0$$

From power series to Taylor series

Observation: Given power series $f(x_0+h) = \sum_{n=0}^{\infty} a_n h^n$

Let's take its derivative:

$$f'(x_0+h) = (a_0 + a_1 h + a_2 h^2 + \cdots)'$$

$$= a_1 + 2a_2 h + 3a_3 h^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n \cdot a_n h^{n-1}$$

$$f''(x_0+h)$$

$$\vdots$$

$$f^{(k)}(x_0+h) = \sum_{n=k}^{\infty} a_n \left(n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)\right) h^{n-k}$$

In particular we have

$$f^{(k)}(x_0) = a_k \cdot k! \Rightarrow \boxed{a_k = \frac{f^{(k)}(x_0)}{k!}}$$

Theorem: Let $f(x_0+h) = \sum_{n=0}^{\infty} a_n h^n$ with $r>0$. Then for h with $|h|<r$, we have.

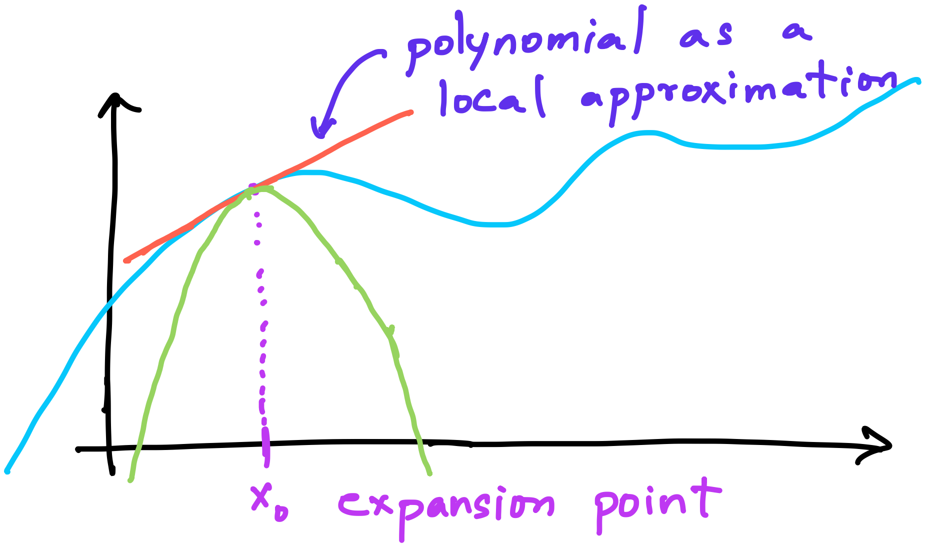
$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n$$

Intuition: start with a power series that converges. Then we have a nice formula that expresses the co-efficients in terms of the derivatives of the function.

Question: Does the theorem hold the other way around? That is, given any func. (possibly with nice assumptions), can we simply build series $\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n}$ and "hope" that it converges to the func. $f(x)$?

Taylor Series

Intuition of Taylor's Theorem:



Linear approximation: $f(x_0+h) = f(x_0) + f'(x_0)\cdot h + r(h)\cdot h$

$(x = x_0 + h)$

with $r(h) \xrightarrow{h\to 0} 0$

Quadratic approximation: $f(x_0+h) = f(x_0) + f'(x_0)\cdot h + \frac{1}{2} f''(x_0)\cdot h^2 + r(h)\cdot h^2$

with $r(h) \xrightarrow{h\to 0} 0$.

Theorem: $I \subset \mathbb{R}$ open interval, $f: I \to \mathbb{R}$.

$f \in \mathcal{C}^{n+1}([a,b])$, $x_0 \in I$. Define

$$T_n(x_0, h) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k$$

→ Taylor series up to degree n

$$R_n(x_0, h) := \int_{x_0}^{x_0+h} \frac{(x+h-t)^n}{n!} f^{(n+1)}(t)\, dt$$

↳ Remainder term.

Then $f(x_0+h) = T_n(x_0,h) + R_n(x_0,h)$

Proof Sketch: follows from Fundamental theorem of calculus, by induction on 'n':

Base case $n=0$ need to prove

$$f(x_0+h) = f(x_0) + \int_{x_0}^{x_0+h} f'(t)\,dt$$

$\cong$ Fundamental theorem of calculus. $\left[\int_a^b F'(x)\,dx = F(b) - F(a) \right]$

Inductive step $n \to n+1$:

- Consider $F(x_0+h) = \frac{(x_0+h-t)^{n+1}}{(n+1)!} f^{(n+1)}(t)$

- Take its derivative
- Integrate and exploit fundemental theorem.

$\boxed{///}$

Theorem (Taylor with Lagrange remainder)

$I \subset \mathbb{R}$, $f: I \to \mathbb{R}$, $f \in C^{n+1}(I)$, $x_0 \in I$

If $h \in \mathbb{R}$ such that $x_0 + h \in I$, then:

$$f(x_0+h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{\text{n-th order Taylor polynomial}} + \underbrace{R_n(h)}_{\text{remainder term}} \quad \text{and}$$

there is ξ with $\xi \in (x_0, x_0+h)$ or $\xi \in (x_0+h, x_0)$

such that $R_n(h) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$.

Often write $f(x_0+h) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k + \mathcal{O}(h^{n+1})$

or with $(x = x_0+h)$: $f(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$

$$+ \mathcal{O}\left((x-x_0)^{n+1}\right)$$

Proof: $$F_{n,h}(t) := \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(h+x_0-t)^k$$

Note: $F_{n,h}(x_0) = T_n(x_0,h)$, $F_{n,h}(x_0+h) = f(x_0+h)$

$$g_{n,h}(t) := (h+x_0-t)^{n+1}, \quad g'_{n,h}(t) = -(n+1)\cdot(h+x_0-t)^n$$

Generalized mean value theorem: $$\frac{F_{n,h}(x_0+h)-F_{n,h}(x_0)}{g_{n,h}(x_0+h)-g_{n,h}(x_0)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

$\xi \in (x_0, x_0+h)$

$$f(x_0+h) - T_n(x_0,h) = \Big(\overbrace{g_{n,h}(x_0+h)}^{0} - \overbrace{g_{n,h}(x_0)}^{h^{n+1}}\Big)\cdot \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

$$= \frac{h^{n+1}\cdot F'_{n,h}(\xi)}{(n+1)\cdot(h+x_0-\xi)^n}$$

$$= \frac{h^{n+1}\cdot \frac{f^{(n+1)}(\xi)}{n!}\cdot \cancel{(h+x_0-\xi)^n}}{(n+1)\cdot\cancel{(h+x_0-\xi)^n}}$$

$$= \frac{h^{n+1}\cdot f^{(n+1)}(\xi)}{(n+1)!}$$

$$F'_{n,h}(t) = \frac{d}{dt}\sum_{k=0}^{n}\frac{f^{(k)}(t)}{k!}\cdot(h+x_0-t)^k$$

$$= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(h+x_0-t)^k - \sum_{k=1}^{n}\frac{f^{(k)}(t)}{(k-1)!}\cdot(h+x_0-t)^{k-1}$$

$$= \frac{f^{(n+1)}(t)}{n!}\cdot(h+x_0-t)^n$$

$$f(x_0+h) = T_n(x_0+h) + \frac{h^{n+1} \cdot f^{(n+1)}(\xi)}{(n+1)!}$$

$\blacksquare$

Theorem: $f \in \mathcal{C}^\infty(I)$, $x_0 \in I$, $h \in \mathbb{R}$ such that $x_0+h \in I$. Define

$$T(x_0,h) := \lim_{n\to\infty} T_n(x_0,h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot h^n .$$

Then we have $f(x) = T(x)$ if $R_n(x_0,h) \xrightarrow{n\to\infty} 0$

For example, this is the case if there exists constants $\alpha, C > 0$ such that

sufficient but not necessary condition.

$$\left| f^{(n)}(t) \right| \leq \alpha \cdot C^n , \quad \forall t \in I, \forall n \in \mathbb{N}.$$

Follows directly from the Lagrangian remainder.

Examples:

- Exponential series:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

power series with $r=\infty$. exp always coincides with its Taylor series.

other examples are sin, cos, polynomials, power series (analytic functions).

- $f(x) = \log(x+1)$, Taylor series around zero.

can prove: convergence radius for Taylor series is $r=1$. For x outside of $(-1,1)$, the Taylor series does not make sense at all.

- $$f(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Has the weird property that $\forall n \in \mathbb{N}$:

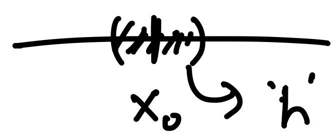
$$f^{(n)}(0) = 0$$

Consider the Taylor series about $x_0 = 0$.

All terms will be zero, i.e. $\forall n : T_n(0, h) = 0$

and $r = \infty$.

$$f(x_0 + h) = T_n(x_0, h) + R_n(x_0, h)$$



$$\underbrace{T_n(x_0 = 0, h) = 0}_{\text{Taylor series around } x_0 = 0 \text{ is zero}} \quad \text{but} \quad \underbrace{f(0 + h) = \exp(-1/h^2)}_{\text{func. value around } x_0 = 0 \text{ is not zero.}}$$

$$\forall (x_0 + h) \neq 0 \quad T_n(x_0, h) \neq f(x_0 + h)$$