

Power Series

Def: A series of the form $p(x) := \sum_{n=0}^{\infty} a_n x^n$

is called a power series.

↳ infinite sum is called a series.
↳ powers of 'x'

Theorem: (Radius of convergence)

For every power series $p(x) := \sum_{n=0}^{\infty} a_n x^n$ there

exists a constant r , $0 \leq r \leq \infty$, called

the radius of convergence such that

• The series converges (absolutely) for all x with $|x| < r$ (means that $\sum_{n=0}^{\infty} a_n |x|^n$ converges, the sequence of partial sums

$P_N(x) := \sum_{n=0}^N a_n |x|^n$ converges "in the usual sense" as $N \rightarrow \infty$)

⚠ unclear what happens when $|x| = r$

• If $|x| < r$, the series even converges uniformly.

The radius of convergence only depends on $(a_n)_n$ and can be computed by various formulae.

- $\gamma = \frac{1}{L}$ where $L = \limsup_{n \rightarrow \infty} (|a_n|)^{1/n}$
 - $\gamma = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$
- } if exist

Examples: • $p(x) = \sum_{n=0}^{\infty} \underbrace{n^c}_{\hookrightarrow a_n} x^n$ for some

constant 'c'

$$\gamma = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n^c}{(n+1)^c} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^c = 1$$

Case $c = -1$: $\sum \frac{1}{n} x^n$ has conv. radius $\gamma = 1$

- For $x = +1$, the series diverges.

$$\sum \frac{1}{n} x^n = \sum \frac{1}{n} \cdot 1^n = \sum_{n=0}^{\infty} \frac{1}{n} \rightarrow \infty$$

- For $x = -1$ it converges.

- For $x > 1$ it diverges.

Case $c=0$: $\sum n^c x^n = \sum x^n$ diverges for $|x| = r$. (both $x=+1$ and $x=-1$).

• Exponential Series:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ has } r = \infty$$

$$\text{since } \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1/n!}{1/(n+1)!} = n+1 \rightarrow \infty$$

• $\sum_{n=0}^{\infty} n! x^n$ has $r = 0$

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \rightarrow 0$$

From power series to Taylor series

Observation: Given power series $f(x_0+h) = \sum_{n=0}^{\infty} a_n h^n$

Let's take its derivative:

$$\begin{aligned} f'(x_0+h) &= (a_0 + a_1 h + a_2 h^2 + \dots)' \\ &= a_1 + 2a_2 h + 3a_3 h^2 + \dots \\ &= \sum_{n=1}^{\infty} n \cdot a_n h^{n-1} \end{aligned}$$

$$\begin{array}{c}
 f''(x_0+h) \\
 \vdots \\
 f^{(k)}(x_0+h) = \sum_{n=k}^{\infty} a_n (n \cdot (n-1) \cdot (n-2) \dots (n-k+1)) h^{n-k}
 \end{array}$$

In particular we have

$$f^{(k)}(x_0) = a_k \cdot k! \Rightarrow a_k = \frac{f^{(k)}(x_0)}{k!}$$

Theorem: Let $f(x_0+h) = \sum_{n=0}^{\infty} a_n h^n$ with $r > 0$. Then for h with $|h| < r$, we

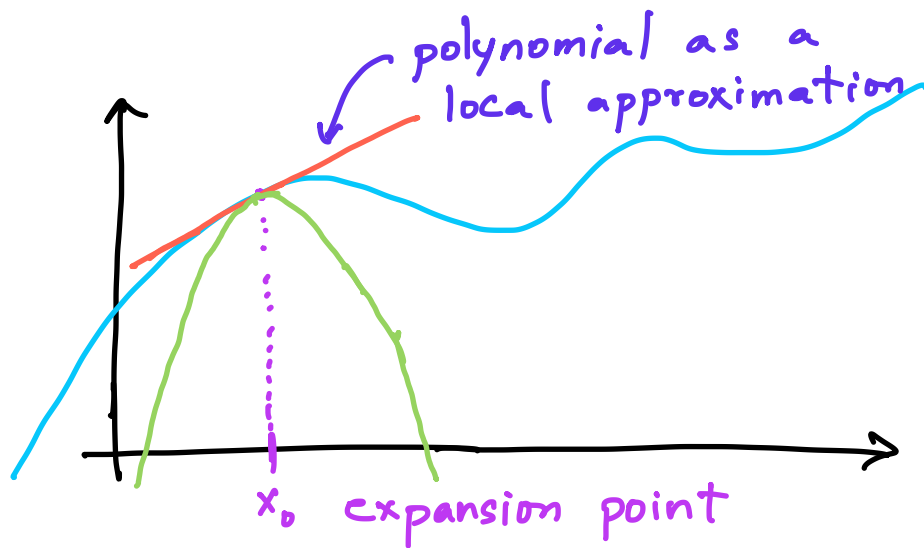
have.
$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n$$

Intuition: start with a power series that converges. Then we have a nice formula that expresses the coefficients in terms of the derivatives of the function.

Question: Does the theorem hold the other way around? That is, given any func. (possibly with nice assumptions)

Taylor Series

Intuition of Taylor's Theorem:



Linear approximation: $f(x_0+h) = f(x_0) + f'(x_0) \cdot h + r(h) \cdot h$
($x = x_0+h$) with $r(h) \xrightarrow{h \rightarrow 0} 0$

Quadratic approximation: $f(x_0+h) = f(x_0) + f'(x_0) \cdot h + \frac{1}{2} f''(x_0) \cdot h^2 + r(h) \cdot h^2$
with $r(h) \xrightarrow{h \rightarrow 0} 0$.

Theorem: $I \subset \mathbb{R}$ open interval, $f: I \rightarrow \mathbb{R}$.

$f \in \mathcal{C}^{n+1}([a, b])$, $x_0 \in I$. Define

$$T_n(x_0, h) := \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \cdot h^k \rightarrow \text{Taylor series up to degree } n$$

- Take its derivative
- Integrate and exploit fundamental theorem.



Theorem (Taylor with Lagrange remainder)

$I \subset \mathbb{R}$, $f: I \rightarrow \mathbb{R}$, $f \in \mathcal{C}^{n+1}(I)$, $x_0 \in I$

If $h \in \mathbb{R}$ such that $x_0 + h \in I$, then:

$$f(x_0 + h) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{\substack{\text{n-th order} \\ \text{Taylor polynomial}}} + \underbrace{R_n(h)}_{\substack{\text{remainder} \\ \text{term}}}$$

there is ξ with $\xi \in (x_0, x_0 + h)$ or $\xi \in (x_0 + h, x_0)$

$$\text{such that } R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$$

$$\text{Often write } f(x_0 + h) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \cdot h^k + \mathcal{O}(h^{n+1})$$

$$\text{or with } (x = x_0 + h): f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \mathcal{O}((x - x_0)^{n+1})$$

Proof: $F_{n,h}(t) := \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (h+x_0-t)^k$

Note: $F_{n,h}(x_0) = T_n(x_0, h)$, $F_{n,h}(x_0+h) = f(x_0+h)$

$g_{n,h}(t) := (h+x_0-t)^{n+1}$, $g'_{n,h}(t) = -(n+1) \cdot (h+x_0-t)^n$

Generalized mean value theorem: $\frac{F_{n,h}(x_0+h) - F_{n,h}(x_0)}{g_{n,h}(x_0+h) - g_{n,h}(x_0)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$

$\xi \in (x_0, x_0+h)$

$f(x_0+h) - T_n(x_0, h) = \left(\overbrace{g_{n,h}(x_0+h)}^0 - \overbrace{g_{n,h}(x_0)}^{h^{n+1}} \right) \cdot \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$

$F'_{n,h}(t) = \frac{d}{dt} \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} \cdot (h+x_0-t)^k$
 $= \sum_{k=0}^n \frac{f^{(k+1)}(t)}{k!} (h+x_0-t)^k$
 $- \sum_{k=1}^n \frac{f^{(k)}(t)}{(k-1)!} \cdot (h+x_0-t)^{k-1}$
 $= \frac{f^{(n+1)}(t)}{n!} \cdot (h+x_0-t)^n$

$= \frac{h^{n+1} \cdot F'_{n,h}(\xi)}{(n+1) \cdot (h+x_0-\xi)^n}$
 $= \frac{h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot \cancel{(h+x_0-\xi)^n}}{(n+1) \cdot \cancel{(h+x_0-\xi)^n}}$
 $= \frac{h^{n+1} \cdot f^{(n+1)}(\xi)}{(n+1)!}$

$$f(x_0+h) = T_n(x_0+h) + \frac{h^{n+1} \cdot f^{(n+1)}(\xi)}{(n+1)!}$$



Theorem: $f \in C^\infty(I)$, $x_0 \in I$, $h \in \mathbb{R}$ such that $x_0+h \in I$. Define

$$T(x_0, h) := \lim_{n \rightarrow \infty} T_n(x_0, h) = \sum_{h=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot h^n.$$

Then we have $f(x) = T(x)$ if $R_n(x_0, h) \xrightarrow{n \rightarrow \infty} 0$

For example, this is the case if there exists

constants $\alpha, C > 0$ such that

sufficient but not necessary condition.

$$|f^{(n)}(t)| \leq \alpha \cdot C^n, \quad \forall t \in I, \quad \forall n \in \mathbb{N}.$$

Follows directly from the Lagrangian remainder.

Examples:

- Exponential series:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

power series with $r = \infty$.
exp always coincides with
its Taylor series.

other examples are sin, cos, polynomials,
power series (analytic functions).

- $f(x) = \log(x+1)$, Taylor series around zero.

can prove: convergence radius for Taylor series

is $r = 1$. For x outside of $(-1, 1)$, the

Taylor series does not make sense at all.

- $f(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Has the weird property that $\forall n \in \mathbb{N}$:

$$f^{(n)}(0) = 0$$

Consider the Taylor series about $x_0 = 0$.

All terms will be zero, i.e. $\forall n: T_n(0, h) = 0$
and $r = \infty$.

$$f(x_0 + h) = T_n(x_0, h) + R_n(x_0, h)$$

$$\frac{(\cancel{x_0})}{x_0 \rightarrow h}$$

$$T_n(x_0=0, h) = 0 \quad \text{but} \quad f(0+h) = \exp(-1/h^2)$$

Taylor series around $x_0=0$
is zero

func. value around
 $x_0=0$ is not zero.

$$\forall (x_0+h) \neq 0 \quad T_n(x_0, h) \neq f(x_0+h)$$