Reimann Integral:

\[ f : \mathbb{R} \to \mathbb{R} \]

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{k} \text{vol}(I_k) \cdot f(m_k)
\]

Problems of Reimann integral:

(i) difficult to extend to higher dimensions.

(ii) dependence on continuity.

(iii) limit processes

\[
\lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \int_{a}^{b} \lim_{n \to \infty} f_n(x) \, dx
\]

Lebesgue Integrals.
Let $X$ be set, $P(X)$ power set of $X$.

Example: $X = \{a, b\}$, $P(X) = \{\emptyset, X, \{a\}, \{b\}\}$

**Def:** $\mathcal{A} \subseteq P(X)$ is called a $\sigma$-algebra:

(a) $\emptyset, X \in \mathcal{A}$

(b) $A \in \mathcal{A} \Rightarrow A^c := X \setminus A \in \mathcal{A}$

(c) $A_i \in \mathcal{A}, i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

**Def:** A measurable space consists of a set $X$ and a $\sigma$-algebra $\mathcal{A}$ over $X$. Notation: $(X, \mathcal{A})$. The sets $A \in \mathcal{A}$ are called $\mathcal{A}$-measurable sets.

**Examples:**

1. $\mathcal{A} = \{\emptyset, X\}$ \Rightarrow smallest

2. $\mathcal{A} = P(X)$ \Rightarrow largest
Let $\mathcal{A}_i$ be a $\sigma$-algebra on $X$, $i \in I$ (index set).

Then $\bigcap_{i \in I} \mathcal{A}_i$ is also a $\sigma$-algebra on $X$.

**Def:** For $\mathcal{M} \subseteq \mathcal{P}(X)$, there is a smallest $\sigma$-algebra that contains $\mathcal{M}$:

$$\sigma(\mathcal{M}) := \bigcap \{A \subseteq M \mid A \text{ is a } \sigma\text{-algebra}\}$$

**Example:** $X = \{a, b, c, d\}$, $\mathcal{M} = \{\{a\}, \{b\}\}$

$$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$$

**Def:** Let $(X, \mathcal{T})$ be a topological space. (Let $X$ be a metric space)

(Def. $X$ be a subset of $\mathbb{R}^n$)

Let $\mathcal{B}(X)$ be the Borel $\sigma$-algebra on $X$.

($\sigma$-algebra generated by the open sets).

$$\mathcal{B}(X) := \sigma(\mathcal{T})$$
**Measures**

**Def:** Let \((X, \mathcal{A})\) be a measurable space. Consider a map \(\mu: \mathcal{A} \to [0, \infty] = [0, \infty) \cup \{\infty\}\) is called a measure if it satisfies:

(a) \(\mu(\emptyset) = 0\)

(b) Additivity: \(\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)\) with \(A_i \cap A_j = \emptyset\), if \(i \neq j\) for all \(A_i \in \mathcal{A}\).

**Def:** A measurable space \((X, \mathcal{A})\) endowed with a measure \(\mu\) is called a measure space \((X, \mathcal{A}, \mu)\).
Examples: \( X, A = P(X) \)

(a) counting measure: \( \mu(A) := \begin{cases} \# A & A \text{ has finitely many elements} \\ \infty, & \text{else} \end{cases} \)

Calculation rules in \([0, \infty] \):

\( x + \infty := \infty \quad \forall x \in [0, \infty] \)

\( x \cdot \infty := \infty \quad \forall x \in (0, \infty] \)

\( 0 \cdot \infty := 0 \) (\! in most cases in measure theory!)

(b) Dirac measure for \( p \in X \)

\[ \delta_p (A) := \begin{cases} 1, & p \in A \\ 0, & \text{else} \end{cases} \]

(c) We want to define a measure on \( X = \mathbb{R}^n \)

1. \( \mu([0,1]^n) = 1 \)

Lebesgue measure

2. \( \mu(x + A) = \mu(A) \quad \forall x \in \mathbb{R}^n \)

(\( \mathcal{F} \)-algebra \( \neq \) power set)
(d) A more useful class of measures on $\mathbb{R}$. Let $F: \mathbb{R} \to \mathbb{R}$ be a monotonically increasing, continuous.

Define a measure $\mu_F$ on $(\mathbb{R}, \mathcal{A})$ by setting $\mu_F(S) = \inf \left\{ \sum_{j=1}^{\infty} F(b_j) - F(a_j) \mid S \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}$

- Cover $S$ by intervals.
- To each interval we assign "elementary volume" $F(b) - F(a)$.
- Take "best" covering.

\[ \triangledown \text{ Need to prove: this is a measure!} \]
A subset $N \in \mathcal{A}$ is called a **null set** if $\mu(N) = 0$. We say that a property holds **almost everywhere** if it holds for all $x \in X$ except for $x$ in a null set $N$.

(in probability theory, we say “almost surely”).