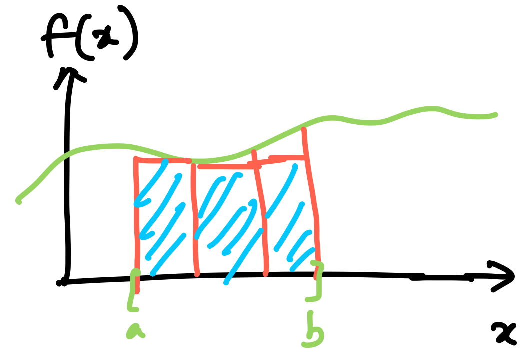
σ- Algebra

Reimann Integral:



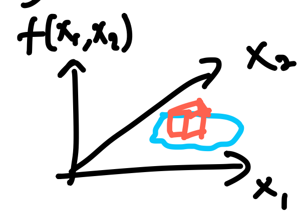
$f : \mathbb{R} \to \mathbb{R}$

$$\int_a^b f\,dt \approx \sum_k \underbrace{vol(I_k)}_{} \cdot f(m_k)$$

$\hookrightarrow x_{k+1} - x_k$

Problems of Reimann integral:

(i) difficult to extend to higher dimensions.

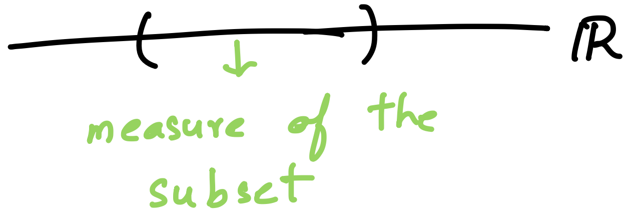


(ii) dependence on continuity.

(iii) limit processes

$$\lim_{n\to\infty} \int_a^b f_n(x)dx \stackrel{?}{=} \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

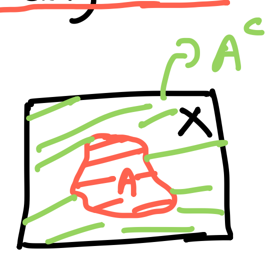
Lebesgue Integrals.



Let X be set, $P(X)$ power set of X.

Example: $X = \{a, b\}$, $P(X) = \{\phi, X, \{a\}, \{b\}\}$

Def: $\mathcal{A} \subseteq P(X)$ is called a σ-algebra:



(a) $\phi, X \in \mathcal{A}$

(b) $A \in \mathcal{A} \Rightarrow A^c := X \setminus A \in \mathcal{A}$

(c) $A_i \in \mathcal{A}$, $i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Def: A measurable space consists of a set X and a σ-algebra $\mathcal{A}$ over X. Notation: $(X, \mathcal{A})$.
The sets $A \in \mathcal{A}$ are called $\mathcal{A}$-measurable sets.

Examples: (1) $\mathcal{A} = \{\phi, X\}$ → smallest

(2) $\mathcal{A} = P(X)$ → largest

Let $\mathcal{A}_i$ be a σ-algebra on X, $i \in I$ (index set)

Then $\bigcap_{i \in I} \mathcal{A}_i$ is also a σ-algebra on X.

Def: For $\mathcal{M} \subseteq P(X)$, there is a smallest σ-algebra that contains $\mathcal{M}$:

$$\sigma(\mathcal{M}) := \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{M} \\ \mathcal{A}\ \sigma\text{-algebras}}} \mathcal{A}$$

← σ-algebra generated by $\mathcal{M}$.

Example: $X = \{a, b, c, d\}$, $\mathcal{M} = \{\{a\}, \{b\}\}$

$$\sigma(\mathcal{M}) = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}, \{a,c,d\}, \{c,d\} \}$$

Def: Let (X, τ) be a topological space.
(Let X be a metric space)
(Let X be a subset of $\mathbb{R}^n$)

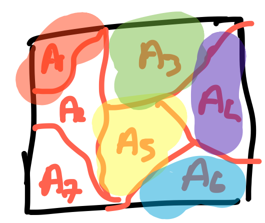
We need "open sets".

$B(X)$ Borel σ-algebra on X.
(the σ-algebra generated by the open sets).

$$B(X) := \sigma(\tau)$$

Measures

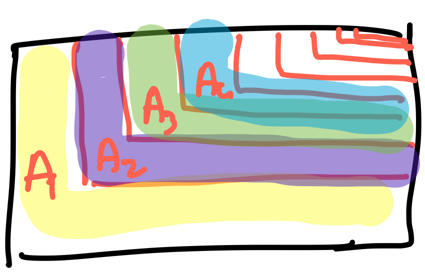
Def: Let $(X, \mathcal{A})$ be a measurable space.
Consider a map $\mu : \mathcal{A} \to [0, \infty]$ $= [0,\infty) \cup \{\infty\}$ is called a
measure if it satisfies:



(a) $\mu(\emptyset) = 0$

additivity (b) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ with $A_i \cap A_j = \emptyset$,

$i \neq j$ for all $A_i \in \mathcal{A}$.



sequence $(A_1, A_2, A_3 \ldots)$

Def: A measurable space $(X, \mathcal{A})$ endowed with a measure μ is called a **measure space** $(X, \mathcal{A}, \mu)$.

Examples: X, $\mathcal{A} = \mathcal{P}(X)$

(a) counting measure: $\mu(A) := \begin{cases} \underbrace{\#A}_{\text{number of elements in } A}, & A \text{ has finitely many elements} \\ \infty, & \text{else} \end{cases}$
$(A \in \mathcal{A})$

Calculation rules in $[0, \infty]$:

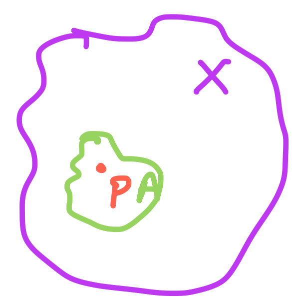
$x + \infty := \infty \quad \forall x \in [0, \infty]$

$x \cdot \infty := \infty \quad \forall x \in (0, \infty]$

$0 \cdot \infty := 0$ (! in most cases in measure theory!)

(b) Dirac measure for $p \in X$

$$\delta_p(A) := \begin{cases} 1, & p \in A \\ 0, & \text{else} \end{cases}$$



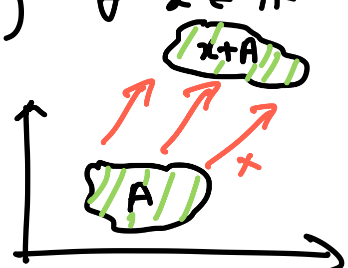
(c) We want to define a measure on $X = \mathbb{R}^n$

(1) $\mu([0,1]^n) = 1$

Lebesgue measure

(2) $\mu(x + A) = \mu(A) \quad \forall x \in \mathbb{R}^n$

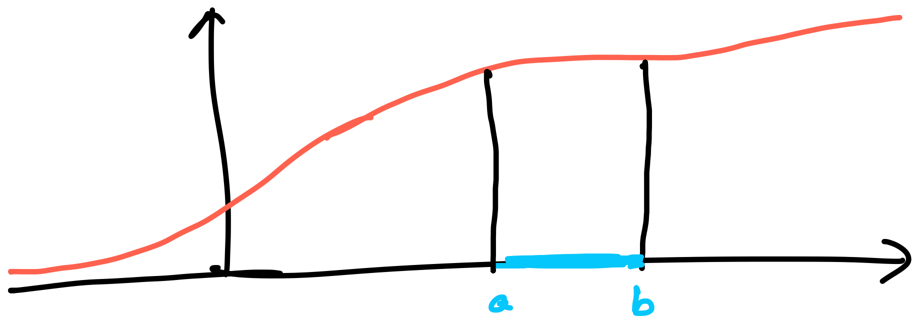
(σ-algebra $\neq$ power set)



(d) A more useful class of measures on $\mathbb{R}$.

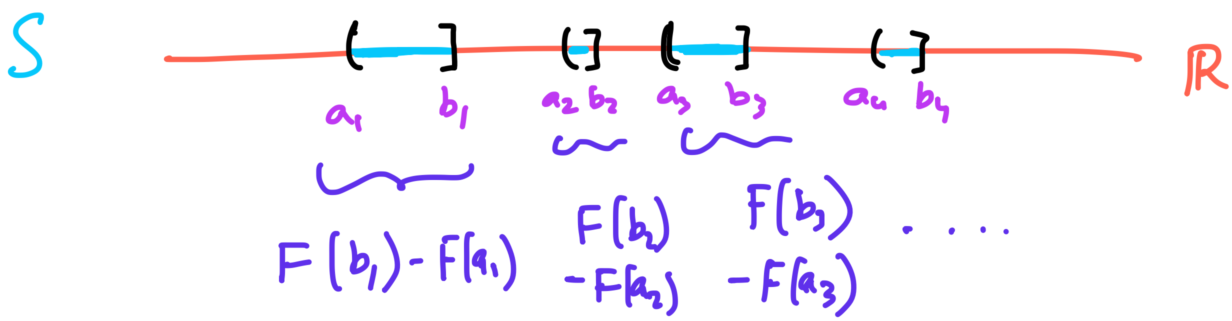
$X = \mathbb{R}$, $\mathcal{A}$ Borel σ-algebra. Let $F: \mathbb{R} \to \mathbb{R}$.

be a monotonically increasing, continuous.



Define a measure μ_F on $(\mathbb{R}, \mathcal{A})$ by

setting $\mu_F(S) = \inf \left\{ \sum_{j=1}^{\infty} F(b_j) - F(a_j) \,\middle|\, S \subset \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$



- cover S by intervals
- To each interval we assign "elementary volume" $F(b) - F(a)$.
- Take "best" covering.

⚠ Need to prove: this is a measure!

A subset $N \in \mathcal{A}$ is called a <u>null set</u> if $\mu(N) = 0$. We say that a property holds <u>almost everywhere</u> if it holds for all $x \in X$ except for x in a null set N.

(in probability theory, we say "almost surely").