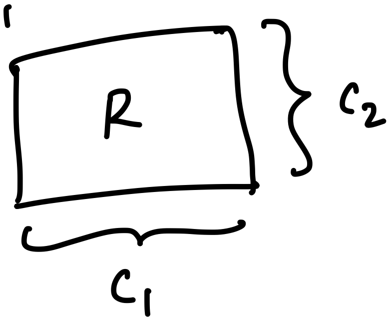
The Lebesgue Measure on $\mathbb{R}^n$

Want to construct a measure on $\mathbb{R}^n$. Want that rectangles of the form $[a_1, b_1) \times [a_2, b_2) \times \dots \times [a_n, b_n)$ have the "natural volume" given by

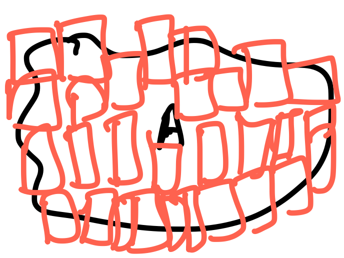
$$\prod_{i=1}^{n} (b_i - a_i)$$



$\longrightarrow \mathrm{vol}(R) := c_1 \cdot c_2$

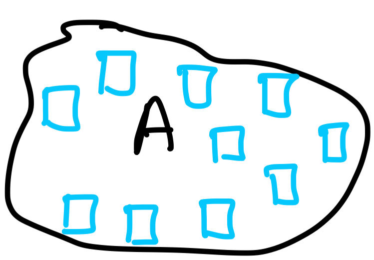
First approaches (Jordan, Reimann) attempted the following:

"outer approximation":



$$A \subset \bigcup_{i=1}^{n} \text{rectangle}_i$$

"inner approximation":

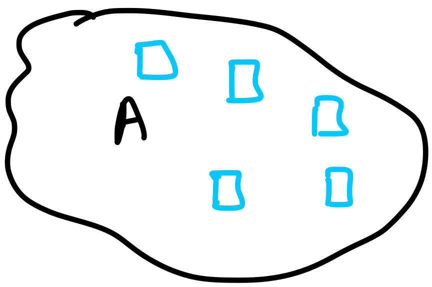


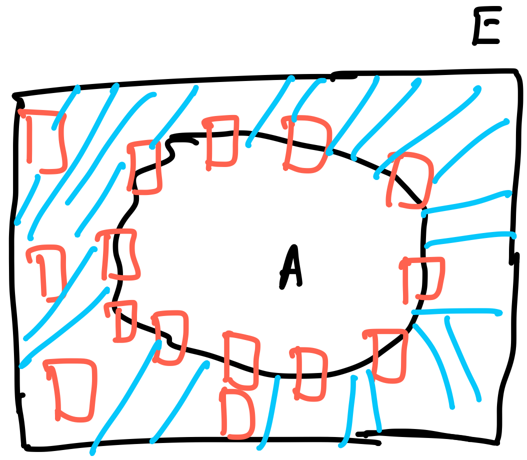
$$\bigcup_{i=1}^{n} \text{rectangle}_i \subset A$$

A would be called "measurable" if outer and inner approximation "converges".

Now we want to generalize this approach:

- Allow for countable coverings.
- Replace inner approximation by an outer approximation of the complement:





outer approx. of $E \setminus A$

$$\mu(E) = \mu(E\setminus A) + \mu(A)$$

$$\Rightarrow \mu(A) = \mu(E) - \underbrace{\mu(E\setminus A)}$$

- Need σ-algebra as underlying structure.

Outer Lebesgue Measure

Set the "natural volume" of rectangles:

$$R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$$

$$|R| := \prod_{i=1}^{n} (b_i - a_i)$$

Definition of outer Lebesgue measure:

Let $A \subset \mathbb{R}^n$ be arbitrary. We define

$$\lambda(A) := \inf \left\{ \sum_{i=1}^{\infty} \underbrace{|R_i|}_{\text{natural volume}} \,\middle|\, A \subset \bigcup_{i=1}^{\infty} R_i ,\ R_i \text{ is rectangle} \right\}$$

We cover A by a countable union of rectangles, then take inf. Observe: $\lambda(A) \in [0,\infty) \cup \{\infty\}$.

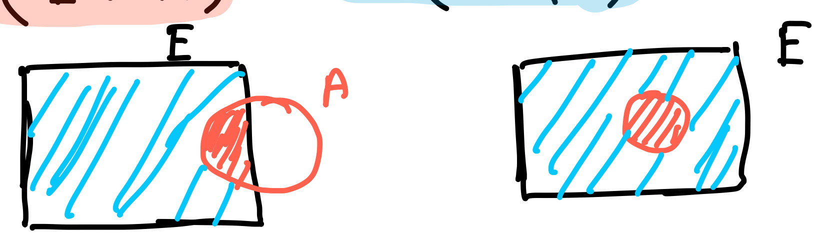
We want to make $\lambda(A)$ into a measure.

Problem: if we use $P(\mathbb{R}^n)$ as σ-algebra, we run into contradictions.

Need to restrict ourselves to a smaller σ-algebra.

Def: We say that a set $A \subset \mathbb{R}^n$ is measurable if $\forall E \subset \mathbb{R}^n$

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \setminus A)$$



Denote by $\mathcal{L}$ all measurable subsets of $\mathbb{R}^n$

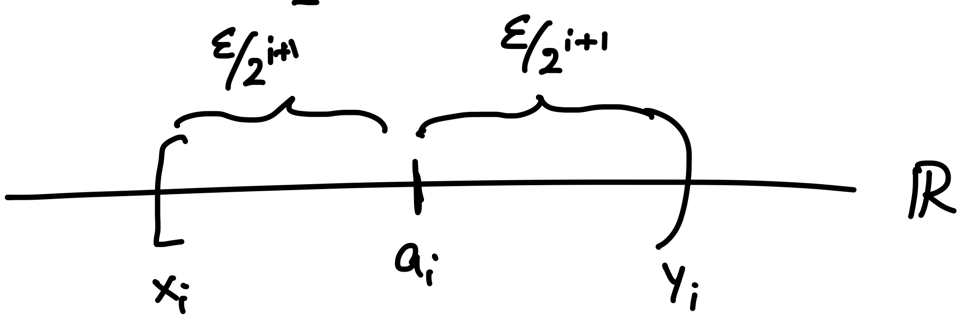
Theorem: The set $\mathcal{L}$ forms a σ-algebra on $\mathbb{R}^n$. The outer measure λ is in fact a measure on $(\mathbb{R}^n, \mathcal{L})$. On rectangles it coincides with the "natural volume".

Examples:

- $\lambda(\{x\}) = 0$
- $\lambda(\mathbb{R}) = \infty$
- $A \subset \mathbb{R}$ coutable. Then $\lambda(A) = 0$. In particular, $\mathbb{Q}$ is measurable and $\lambda(\mathbb{Q}) = 0$.

Proof sketch: For $\varepsilon > 0$, define for all $a_i \in A$ the interval $[x_i, y_i)$ such that

$$x_i = a_i - \frac{\varepsilon}{2^{i+1}}, \quad y_i = a_i + \frac{\varepsilon}{2^{i+1}}$$



$$A \subset \bigcup_{i=1}^{\infty} [x_i, y_i)$$

$$\Rightarrow \lambda(A) \leq \sum_{i=1}^{\infty} \lambda([x_i, y_i))$$

$$= \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \varepsilon$$

Taking the inf. over all coverings shows that

$\lambda(A) = 0$

Comparison of $\mathcal{L}$ (σ-algebra of Lebesgue measurable sets) with the Borel σ-algebra $\mathcal{B}$

(1) $\mathcal{B} \subset \mathcal{L}$:

- open intervals are measurable, thus in $\mathcal{L}$
- any open set A in $\mathbb{R}^n$ can be written as a countable union of open intervals:

  $A \subset \bigcup_{i=1}^{\infty} I_i$, I_i open interval.

(2) For every Lebesgue-measurable set L, there exists a set $B \in \mathcal{B}$ and $N \in \mathcal{L}$ with $\lambda(N) = 0$ such that $L = B \cup N$.

Summary: $\mathcal{L} \approx \mathcal{B}$ (upto sets of measure 0).

A non-measurable set

__Measure problem:__ Search measure μ on $\mathcal{P}(\mathbb{R})$ with the following properties:

(1) $\mu([a,b]) = b-a$, $b > a$

(2) $\mu(x+A) = \mu(A)$, $A \in \mathcal{P}(\mathbb{R})$, $x \in \mathbb{R}$

$\Rightarrow$ μ does not exist.

__Claim:__ Let μ be a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2). $\Rightarrow \mu = 0$.

__Proof:__ (a) __Definitions:__ $I := (0,1]$ with equivalence relation on I.

$$x \sim y \;:\Longleftrightarrow\; x - y \in \mathbb{Q}$$

i.e. $[x] := \{ x + r \mid r \in \mathbb{Q},\ x + r \in I \}$

$$\underset{\downarrow a_1}{\boxed{[x_1]}}\,\underset{\downarrow a_2}{\boxed{[x_2]}}\,\underset{\downarrow a_3}{\boxed{[x_3]}} \;\cdots\cdots = I \qquad a_1\ a_2\ a_3 \ldots$$

> [disjoint decomposition of I into boxes, possibly uncountable many of them!]

$A \subseteq I$ with properties:

(i) For each $[x]$, there is an $a \in A$ with $a \in [x]$

(ii) For all $a, b \in A$: $a, b \in [x] \Rightarrow a = b$.

$A = \{a_1, a_2, \dots\} \rightarrow$ we need axiom of choice of set theory

$A_n := r_n + A$, where $(r_n)_{n \in \mathbb{N}}$ enumeration of $\mathbb{Q} \cap (-1, 1]$.

(b) Claim: $A_n \cap A_m = \emptyset \Leftarrow n \neq m$.

Proof: $x \in A_n \cap A_m \Rightarrow$ $x = r_n + a_n$, $a_n \in A$
$x = r_m + a_m$, $a_m \in A$

$\Rightarrow r_n + a_n = r_m + a_m \Rightarrow a_n - a_m = r_m - r_n \in \mathbb{Q} \Rightarrow a_n \sim a_m$

$\Rightarrow a_n \in [a_m] \Rightarrow a_n = a_m \Rightarrow r_m = r_n \Rightarrow n = m$.

(c) Claim: $(0, 1] \subseteq \bigcup_{n \in \mathbb{N}} A_n \subseteq (-1, 2]$

Proof: exercise for you!

<u>Now assume:</u> μ is a measure on $P(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2).

By (2): $\mu(\underbrace{r_n + A}_{A_n}) = \mu(A) \quad \forall n \in \mathbb{N}$

By (c): $\boxed{\mu((0,1]) \leq \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \mu((-1,2])}$ (*)

<u>We know:</u> $\mu((0,1]) =: C < \infty$

$\mu((-1,2]) = \mu((-1,0] \cup (0,1] \cup (1,2]) = 3C$

(by using (2) and σ-addivity).

$\xRightarrow{(*),(b)} C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C$

$\Rightarrow \boxed{C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3 \cdot C}$

(i) $\mu(A) = 0$, $\sum_{n=1}^{\infty} \mu(A) = 0 \Rightarrow C = 0$

(ii) $\mu(A) > 0$, $\sum_{n=1}^{\infty} \mu(A) = \infty$, $\cancel{C \leq \infty \leq 3C}$

$\Rightarrow \mu(A) = 0$

$$\mu(A) = 0 \Rightarrow C = 0 \quad (\text{hence } \mu((0,1]) = 0)$$

$$\mu(\mathbb{R}) = \mu\Big(\bigcup_{m \in \mathbb{Z}} (m, m+1]\Big) = 0$$

$$\Rightarrow \mu = 0 \qquad \blacksquare$$