The Lebesgue Measure on $\mathbb{R}^n$

Want to construct a measure on $\mathbb{R}^n$. Want that rectangles of the form $[a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$ have the “natural volume” given by

$$\prod_{i=1}^{n} (b_i - a_i)$$

$$\text{vol}(R) := c_1 \cdot c_2$$

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First approaches (Jordan, Reimann) attempted the following:

“outer approximation”:

$$A \subseteq \bigcup_{i=1}^{n} \text{rectangle}_i$$

“inner approximation”:

$$\bigcup_{i=1}^{n} \text{rectangle}_i \subseteq A$$

A would be called “measurable” if outer and inner approximation “converges.”
Now we want to generalize this approach:

- Allow for countable coverings.

- Replace inner approximation by an outer approximation of the complement:

\[ \mu(E) = \mu(E \setminus A) + \mu(A) \]

\[ \Rightarrow \mu(A) = \mu(E) - \mu(E \setminus A) \]

- Need \( \sigma \)-algebra as underlying structure.

\[ \text{Outer Lebesgue Measure} \]

Set the "natural volume" of rectangles:

\[ R = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n \]

\[ |R| := \prod_{i=1}^{n} (b_i - a_i) \]
Definition of outer Lebesgue measure:

Let \( A \subseteq \mathbb{R}^n \) be arbitrary. We define

\[
\lambda(A) := \inf \left\{ \sum_{i=1}^{\infty} |R_i| \mid A \subseteq \bigcup_{i=1}^{\infty} R_i, \text{ } R_i \text{ is rectangle} \right\}
\]

We cover \( A \) by a countable union of rectangles, then take inf. Observe: \( \lambda(A) \in [0, \infty) \cup \{\infty\} \).

We want to make \( \lambda(A) \) into a measure.

Problem: if we use \( \mathcal{P}(\mathbb{R}^n) \) as \( \sigma \)-algebra, we run into contradictions.

Need to restrict ourselves to a smaller \( \sigma \)-algebra.

**Def:** We say that a set \( A \subseteq \mathbb{R}^n \) is **measurable** if \( \forall E \subseteq \mathbb{R}^n \)

\[
\lambda(E) = \lambda(E \cap A) + \lambda(E \backslash A)
\]

Denote by \( \mathcal{L} \) all measurable subsets of \( \mathbb{R}^n \).
Theorem: The set $\mathcal{L}$ forms a $\sigma$-algebra on $\mathbb{R}$.

The outer measure $\lambda$ is in fact a measure on $(\mathbb{R}^n, \mathcal{L})$. On rectangles it coincides with the "natural volume".

Examples:

- $\lambda(\{x\}) = 0$
- $\lambda(\mathbb{R}) = \infty$

- A $\subset \mathbb{R}$ countable. Then $\lambda(A) = 0$. In particular, $\emptyset$ is measurable and $\lambda(\emptyset) = 0$.

Proof sketch: For $\varepsilon > 0$, define for all $a_i \in A$ the interval $[x_i, y_i)$ such that

$$x_i = a_i - \frac{\varepsilon}{2^{i+1}}, \quad y_i = a_i + \frac{\varepsilon}{2^{i+1}}$$

$$\bigcup_{i=1}^{\infty} [x_i, y_i)$$

$A \subset \bigcup_{i=1}^{\infty} [x_i, y_i)$
\[ \lambda(A) \leq \sum_{i=1}^{\infty} \lambda\left(\bigcup_{i} (x_i, y_i)\right) \]

\[ = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \varepsilon \]

Taking the inf. over all coverings shows that

\[ \lambda(A) = 0 \]

Comparison of \( \mathcal{L} \) (\( \sigma \)-algebra of Lebesgue measurable sets) with the Borel \( \sigma \)-algebra \( \mathcal{B} \)

1. \( \mathcal{B} \subset \mathcal{L} \):
   - open intervals are measurable, thus in \( \mathcal{L} \)
   - any open set \( A \) in \( \mathbb{R}^n \) can be written as a countable union of open intervals:
     \[ A \subset \bigcup_{i=1}^{\infty} I_i, \ I_i \text{ open interval}. \]

2. For every Lebesgue-measurable set \( L \), there exists a set \( B \in \mathcal{B} \) and \( N \in \mathcal{L} \) with \( \lambda(N) = 0 \) such that \( L = B \cup N \).

Summary: \( \mathcal{L} \approx \mathcal{B} \) (up to sets of measure 0).
A non-measurable set

Measure problem: Search measure $\mu$ on $P(\mathbb{R})$ with the following properties:

(1) $\mu([a, b]) = b - a$, $b > a$

(2) $\mu(x + A) = \mu(A)$, $A \in P(\mathbb{R})$, $x \in \mathbb{R}$

$\Rightarrow \mu$ does not exist.

Claim: Let $\mu$ be a measure on $P(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2). $\Rightarrow \mu = 0$.

Proof: (a) Definitions: $I := (0, 1]$ with equivalence relation on $I$.

$x \sim y \iff x - y \in \mathbb{Q}$

i.e. $[x] := \{x + y \mid y \in \mathbb{Q}, x + y \in I\}$

$\left\{ [x_1], [x_2], [x_3], \ldots \right\} \sim I$ [disjoint decomposition of $I$ into boxes, possibly uncountable many of them]
$A \subseteq I$ with properties:

(i) For each $[x]$, there is an $a \in A$ with $a \in [x]$.

(ii) For all $a, b \in A : a, b \in [x] \Rightarrow a = b$.

$A = \{a_1, a_2, \ldots \} \Rightarrow$ we need axiom of choice of set theory.

$A_n := \tau_n + A$, where $(\tau_n)_{n \in \mathbb{N}}$ enumeration of $\mathbb{Q} \cap (-1, 1]$.

(b) **Claim:** $A_n \cap A_m = \emptyset \iff n \neq m$.

**Proof:** $x \in A_n \cap A_m \Rightarrow x = \tau_n + a_n, a_n \in A$

$x = \tau_m + a_m, a_m \in A$

$\Rightarrow \tau_n + a_n = \tau_m + a_m \Rightarrow a_n - a_m = \tau_m - \tau_n \in \mathbb{Q} \Rightarrow a_n - a_m$.

$\Rightarrow a_n \in [a_m] \Rightarrow a_n = a_m \Rightarrow \tau_m = \tau_n \Rightarrow n = m$.

(c) **Claim:** $(0, 1] \subseteq \bigcup_{n \in \mathbb{N}} A_n \subseteq (-1, 2]$

**Proof:** exercise for you!
Now assume: $\mu$ is a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2).

By (2): $\mu(\cap_{n \in \mathbb{N}} A_n) = \mu(A) \quad \forall \, n \in \mathbb{N}$

By (c): $\mu([0,1]) = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \mu([-1,2]) \quad (*)$

We know: $\mu((0,1]) = : c < \infty$

$\mu([-1,2]) = \mu([-1,0] \cup (0,1] \cup (1,2]) = 3c$
(by using (2) and $\sigma$-additivity).

$(\ast), (b) \quad c \leq \sum_{n=1}^{8} \mu(A_n) \leq 3c$

$\Rightarrow \quad c \leq \sum_{n=1}^{8} \mu(A) \leq 3c$

(i) $\mu(A) = 0$, $\sum_{n=1}^{8} \mu(A) = 0 \Rightarrow c = 0$

(ii) $\mu(A) > 0$, $\sum_{n=1}^{8} \mu(A) = \infty$, $c \leq \infty \leq 3c$

$\Rightarrow \quad \mu(A) = 0$
\[ \mu(A) = 0 \implies C = 0 \quad (\text{hence } \mu((0,1)) = 0) \]

\[ \mu(\mathbb{R}) = \mu \left( \bigcup_{m \in \mathbb{Z}} (m, m+1] \right) = 0 \]

\[ \implies \mu = 0 \]