The Lebesgue Integral on $\mathbb{R}^n$

Intuition:

**Reimann**

- bounded
- continuous
- finite set of rectangles.

**Lebesgue**

- not bounded
- need not be continuous
- countable sets

$f^{-1}(G)$
**Def:** A function \( f : (\Omega_1, A_1) \to (\Omega_2, A_2) \) between two measurable spaces is called a measurable if pre-images of measurable sets are measurable:

\[
\forall A_2 \in A_2 : f^{-1}(A_2) \in A_1
\]

\[
\{ x \in \Omega_1 \mid f(x) \in A_2 \}
\]

\[
\mu( f^{-1}(\{ 1 \}) ) \in A_1
\]

\[
(\Omega_2, A_2), \quad (\mathbb{R}, B(\mathbb{R}))
\]

Characteristic function (also indicator function)

\[
\chi_A : \Omega \to \mathbb{R}, \quad \chi_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}
\]
Def: $\phi: \mathbb{R}^n \to \mathbb{R}$ is called a simple function if there exist measurable sets $A_i \subset \mathbb{R}^n$, $c_i \in \mathbb{R}$ such that

$$\phi(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$

$$I = c_1 \mu(A_1) + c_2 \mu(A_2) + c_3 \mu(A_3)$$

$$\phi(x) = c_1 \chi_{A_1}(x) + c_2 \chi_{A_2}(x) + c_3 \chi_{A_3}(x)$$

$$I(\phi) = \int \phi \, d\mu = \sum_{i=1}^{n} c_i \mu(A_i) \rightarrow \text{Lebesgue integral for simple func.}$$

Problem: $3 \cdot \infty - 2 \cdot \infty$ ??
For a function $f^+ : \mathbb{R}^n \to [0, \infty)$ we define its Lebesgue integral

$$\int f^+ \, d\mu = \sup \left\{ \int \phi \, d\mu \mid \phi \leq f, \phi \text{ simple} \right\}$$

(might be $\infty$)

For a general function $f : \mathbb{R}^n \to \mathbb{R}$ we split the function into positive and negative parts: $f = f^+ - f^-$, $f^+ \geq 0$, $f^- \geq 0$

where $f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Note: $f^+$, $f^-$ are measurable if $f$ is measurable.
If both \( f^+ \) and \( f^- \) satisfy \( \int f^+ d\mu < \infty \), \( \int f^- d\mu < \infty \), then we call \( f \) integrable and define

\[
\int f d\mu = \int f^+ d\mu - \int f^- d\mu
\]

Much more powerful notion than Reimann Integral

Example: \( \int \chi_\Omega d\mu = 1 \cdot \mu(\Omega) = 0 \)

Two important Theorems

Theorem (monotone convergence): Consider a sequence of functions \( f_n : \mathbb{R}^n \rightarrow [0, \infty) \) that is pointwise non-decreasing: \( \forall x \in \mathbb{R}^n, f_{k+1}(x) \geq f_k(x) \).

Assume that all \( f_k \) are measurable, and that the pointwise limit exists

\( \forall x : \lim f_k(x) =: f(x) \)
Then:
\[
\int \lim_{k \to \infty} f_k(x) \, dx = \lim_{k \to \infty} \int f_k(x) \, dx
\]
\[
\downarrow
\]
\[
\int f(x) \, dx
\]

**Theorem (dominated convergence):**

Let \( f_k : B \to \mathbb{R} \), \( |f_k(x)| \leq g(x) \) on \( B \), \( g(x) \) is integrable. Assume that the pointwise limit exists: \( \forall x \in B \), \( f(x) := \lim_{n \to \infty} f_n(x) \).

Then:
\[
\int \lim_{k \to \infty} f_k(x) \, dx = \lim_{k \to \infty} \int f_k(x) \, dx
\]
\[
\downarrow
\]
\[
\int f(x) \, dx
\]
Partial Derivatives on $\mathbb{R}^n$

Consider $f: \mathbb{R}^n \rightarrow \mathbb{R}$

**Def:** $f$ is called partially differentiable with respect to variable $x_j$ at point $\xi \in \mathbb{R}^n$ if the function

$$x_j \mapsto g(x_j) := f(\xi_1, \xi_2, \ldots, \xi_{j-1}, x_j, \xi_{j+1}, \ldots, \xi_n)$$

$g: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $\xi_j \in \mathbb{R}$.  

The $j$-th unit vector is \( j^{\text{th}} \text{ unit vector} = \left( \begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array} \right) \).

**Notation:**  

$$\frac{\partial f(\xi)}{\partial x_j} = \lim_{h \to 0} \frac{f(\xi + e_j \cdot h) - f(\xi)}{h}$$

$\mathbb{R}^n \Rightarrow x = \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right)$,  

$f(x) = x_1^2 + x_2^2 \cdot x_1$  

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$

If all partial derivatives exist, then the vector of all partial derivatives is called the **gradient**:

$$\text{grad}(f)(\xi) = \nabla f(\xi) = \left( \begin{array}{c} \frac{\partial f(\xi)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\xi)}{\partial x_n} \end{array} \right) \in \mathbb{R}^n$$
If \( f: \mathbb{R}^n \to \mathbb{R}^m \), we decompose \( f \) into its \( m \) component functions \( f = (f_1, \ldots, f_m) \). We define the Jacobian matrix:

\[
Df(x) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \ldots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix} \in \mathbb{R}^{m \times n}
\]

\( \nabla f_1(x) \) \n\( \nabla f_m(x) \)

⚠️ Even if all partial derivatives exist at \( \mathbf{x} \), we do not know if \( f \) is continuous at \( \mathbf{x} \).

**Example:** \( f: \mathbb{R}^2 \to \mathbb{R} \),

\[
f(x,y) = \begin{cases} 
\frac{x \cdot y}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\
0, & \text{if } x = y = 0
\end{cases}
\]

For \( (x,y) \neq (0,0) \)

\[
\nabla f(x,y) = \left( y \cdot \frac{y^2 - x^2}{(x^2 + y^2)^2}, \ x \cdot \frac{x^2 - y^2}{(x^2 + y^2)^2} \right)
\]

\( \nabla f(0,0) = 0 \) since \( f(x,0) = 0 \ \forall x \) and \( f(0,y) = 0 \ \forall y \) but \( f \) is not continuous at \( 0 \).
Total Derivative

$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\xi \in U$

$f$ is differentiable at $\xi$ if there exists a linear mapping $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that for $h \in \mathbb{R}^n$,

$$f(\xi + h) - f(\xi) = L(h) + \epsilon(h)$$

with

$$\lim_{{h \to 0}} \frac{\epsilon(h)}{|h|} \rightarrow 0.$$
Theorem: \( f : \mathbb{R}^n \to \mathbb{R} \) differentiable at \( \xi \)

- Then \( f \) is continuous at \( \xi \)
- The linear functional \( L \) coincides with the gradient:

\[
f(\xi + h) - f(\xi) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\xi) \cdot h_j + \sigma(h)
\]

\[
= \langle \nabla f(\xi), h \rangle + \sigma(h)
\]

If \( f : \mathbb{R}^n \to \mathbb{R}^m \), it is differentiable if all coordinate functions \( f_1, f_2, \ldots, f_m \) are differentiable. Then all partial derivatives exist and \( L(h) = (\text{Jacobian matrix}) \cdot h \)

Theorem: If all partial derivatives exist and are all continuous, then \( f \) is differentiable.

⚠️ If partial derivatives exist, but are not continuous, then \( f \) doesn't need to be differentiable.
**Directional Derivatives**

**Def:** Assume $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable, $\mathbf{v} \in \mathbb{R}^n$ with $||\mathbf{v}|| = 1$. The directional derivative of $f$ at $\xi$ in the direction of $\mathbf{v}$ is defined as,

$$D_{\mathbf{v}}f(\xi) = \lim_{t \to 0} \frac{f(\xi + t \cdot \mathbf{v}) - f(\xi)}{t}$$

**Theorem:** $f: \mathbb{R}^n \rightarrow \mathbb{R}$ differentiable at $\xi$. Then all the directional derivatives exist, and we can compute them as:

$$D_{\mathbf{v}}f(\xi) = (\nabla f(\xi))^T \mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial f(\xi)}{\partial x_i}$$

The largest value of all directional derivatives is attained in direction: $\mathbf{v} = \frac{\nabla f(\xi)}{||\nabla f(\xi)||}$