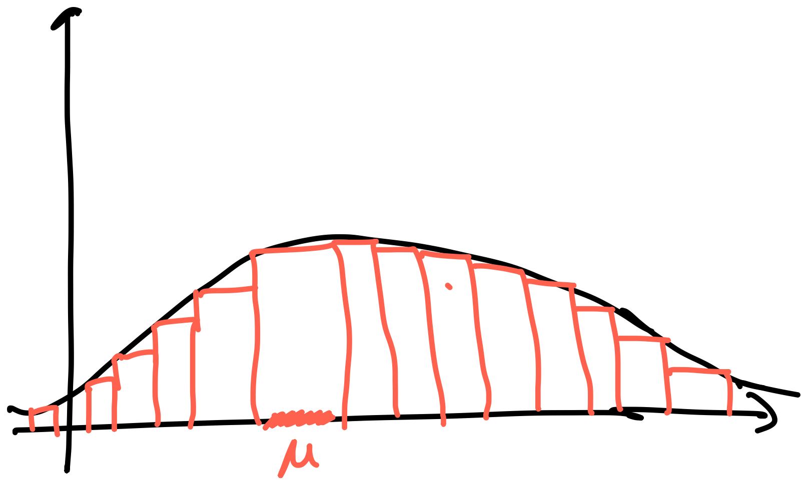
The Lebesgue Integral on $\mathbb{R}^n$

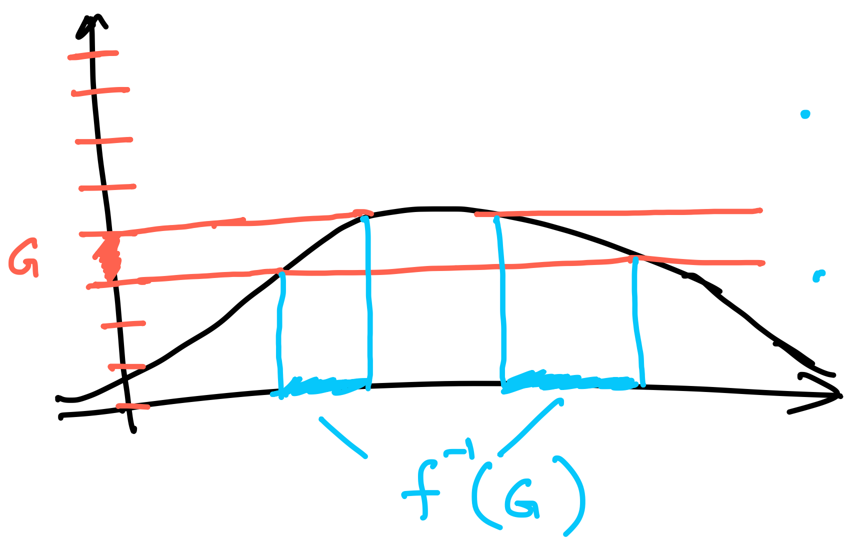
Intuition:

Reimann



- bounded
- continuous
- finite set of rectangles.

Lebsegue

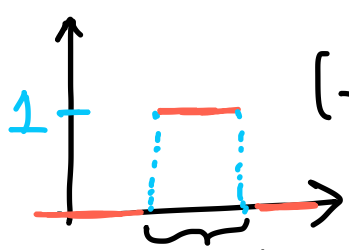


- not bounded
- need not be continuous
- countable sets

Def: A function $f : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ between two measurable spaces is called a <u>measurable</u> if pre-images of measurable sets are measurable:

$$\forall\, A_2 \in \mathcal{A}_2 : f^{-1}(A_2) \in \mathcal{A}_1$$

$$\hookrightarrow =: \{ x \in \Omega_1 \mid f(x) \in A_2 \}$$



$(\Omega, \mathcal{A}, \mu)$

$1 \cdot \mu(A)$

$$\mu(\underbrace{f^{-1}(\{1\})}_{\in \mathcal{A}_1})$$

$(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

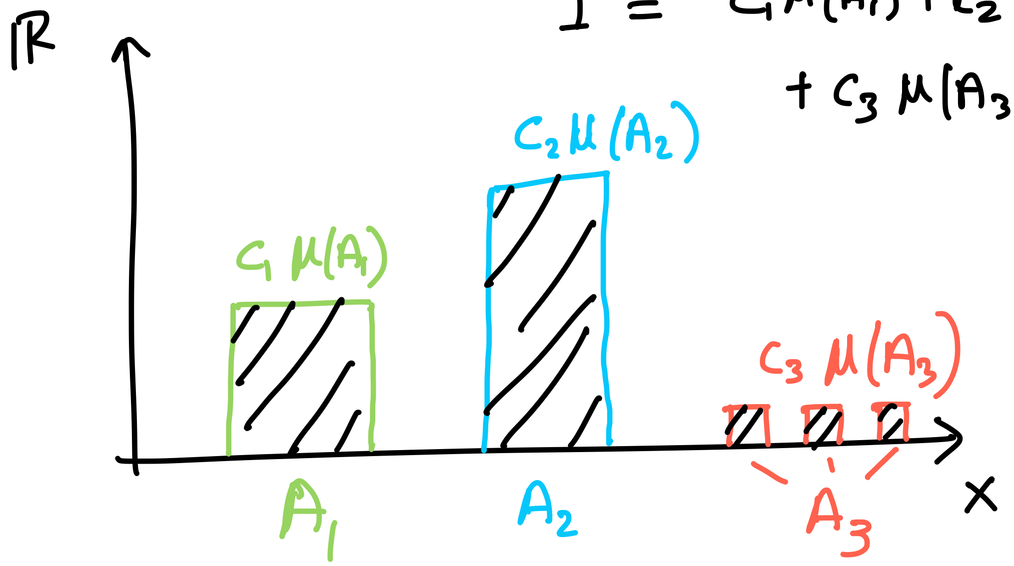
Characteristic function (also indicator func.)

$$\chi_A : \Omega \to \mathbb{R}, \quad \chi_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

Def: $\phi : \mathbb{R}^n \to \mathbb{R}$ is called a simple function if there exist measurable sets $A_i \subset \mathbb{R}^n$, $c_i \in \mathbb{R}$ such that

$$\phi(x) = \sum_{i=1}^{n} c_i \, \chi_{A_i}(x)$$

$$I = c_1 \mu(A_1) + c_2 \mu(A_2) + c_3 \mu(A_3)$$



$$\phi(x) = c_1 \chi_{A_1}(x) + c_2 \chi_{A_2}(x) + c_3 \chi_{A_3}(x)$$

$$I(\phi) = \int \phi \, d\mu = \sum_{i=1}^{n} c_i \, \mu(A_i)$$ → Lebsegue integral for simple func.

Problem: $3 \cdot \infty - 2 \cdot \infty$??

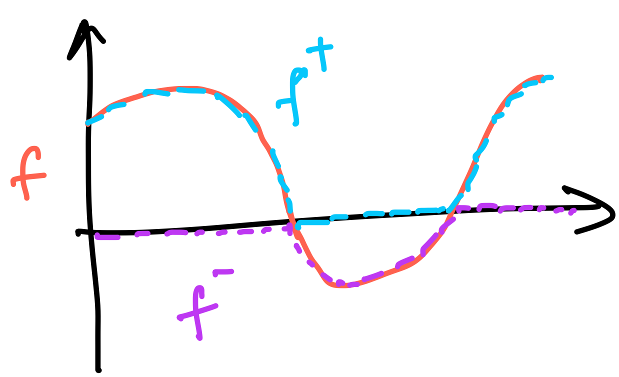
For a function $f^+ : \mathbb{R}^n \to [0, \infty)$ we define its Lebsegue integral

$$\int f^+ d\mu = \sup \left\{ \int \phi \, d\mu \;\middle|\; \phi \leq f, \ \phi \text{ simple} \right\}$$

(might be ∞)

For a general function $f : \mathbb{R}^n \to \mathbb{R}$ we split the function into positive and negative parts: $f = f^+ - f^-$, $f^+ \geq 0$, $f^- \geq 0$

where
$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$f = f^+ - f^-$

<u>Note</u>: f^+, f^- are measurable if f is measurable.

If both f^+ and f^- satisfy $\int f^+ d\mu < \infty$ $\int f^- d\mu < \infty$, then we call f integrable and define

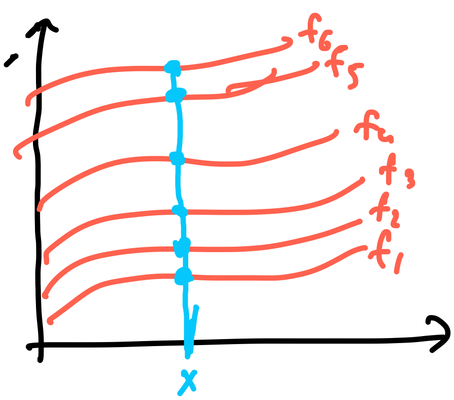
$$\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$$

Much more powerful notion than Reimann Integral.

Example: $\int \chi_{\mathbb{Q}} \, d\mu = 1 \cdot \mu(\mathbb{Q}) = 0$

Two important Theorems

Theorem (monotone convergence): Consider a sequence of functions $f_n : \mathbb{R}^n \to [0, \infty)$ that is pointwise non-decreasing: $\forall x \in \mathbb{R}^n,\ f_{k+1}(x) \geq f_k(x)$.
Assume that all f_k are measurable, and that the pointwise limit exists
$\forall x : \lim f_k(x) =: f(x)$



Then:

$$\int \lim_{k\to\infty} f_k(x)\,dx = \lim_{k\to\infty} \int f_k(x)\,dx$$

$$\Downarrow$$

$$\int f(x)\,dx$$

Theorem (dominated convergence):

$f_k : B \to \mathbb{R}$, $|f_k(x)| \leq g(x)$ on B, $g(x)$ is integrable. Assume that the pointwise limit exists: $\forall x \in B$, $f(x) := \lim_{n\to\infty} f_n(x)$.

Then:

$$\int \lim_{k\to\infty} f_k(x)\,dx = \lim_{k\to\infty} \int f_k(x)\,dx$$

$$\Downarrow$$

$$\int f(x)\,dx$$

Partial Derivatives on $\mathbb{R}^n$

Consider $f: \mathbb{R}^n \to \mathbb{R}$

Def: f is called partially differentiable with respect to varaible x_j at point $\xi \in \mathbb{R}^n$ if the function

$$x_j \mapsto g(x_j) := f(\xi_1, \xi_2, \dots \xi_{j-1}, x_j, \xi_{j+1}, \dots \xi_n)$$

$g: \mathbb{R} \to \mathbb{R}$

is differentiable at $\xi_j \in \mathbb{R}$.

j-th unit vector $= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to j$

Notation: $$\frac{\partial f(\xi)}{\partial x_j} = \lim_{h \to 0} \frac{f(\xi + e_j \cdot h) - f(\xi)}{h}$$

$h \in \mathbb{R}$

$\mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $f(x) = x_1^2 + x_2^2 \cdot x_1$

$f: \mathbb{R}^2 \to \mathbb{R}$

If all partial derivatives exist, then the vector of all partial derivatives is called the gradient:

$$\operatorname{grad}(f)(\xi) = \nabla f(\xi) = \begin{pmatrix} \frac{\partial f(\xi)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\xi)}{\partial x_n} \end{pmatrix} \in \mathbb{R}^n$$

If $f: \mathbb{R}^n \to \mathbb{R}^m$, we decompose f into its m component functions $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$. We define the Jacobian matrix:

$\nabla f_1(x)$

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots\cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$\nabla f_m(x)$

Even if all partial derivatives exist at ξ, we do not know if f is continuous at at ξ.

<u>Example:</u> $f: \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \dfrac{x \cdot y}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

For $(x,y) \neq (0,0)$

$$\nabla f(x,y) = \left(y \cdot \frac{y^2 - x^2}{(x^2+y^2)^2},\ x \cdot \frac{x^2 - y^2}{(x^2+y^2)^2} \right)$$

$\nabla f(0,0) = 0$ since $f(x,0) = 0 \ \forall x$, $f(0,y) = 0 \ \forall y$

but f is not continuous at 0.

Total Derivative

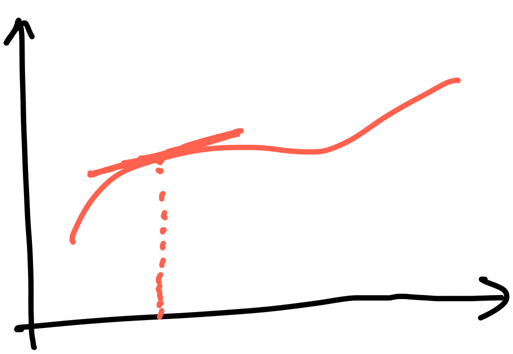
$f: \mathbb{R}^n \to \mathbb{R}^m$, $\xi \in U$

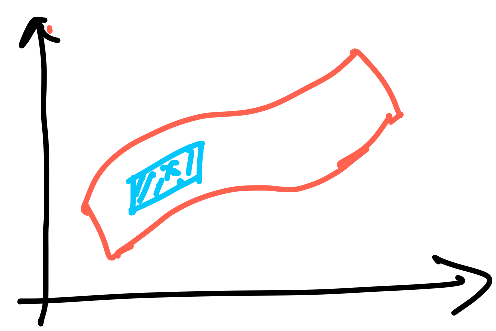
f is differentiable at ξ if there exists a linear mapping $L: \mathbb{R}^n \to \mathbb{R}^m$ such that

for $h \in \mathbb{R}^n$,

$$f(\xi + h) - f(\xi) = L(h) + r(h)$$

with $\lim\limits_{h \to 0} \dfrac{r(h)}{|h|} \to 0.$





Intuition: f is "locally linear"

Theorem: $f : \mathbb{R}^n \to \mathbb{R}$ differentiable at ξ

- Then f is continuous at ξ
- The linear functional L coincides with the gradient:

$$f(\xi + h) - f(\xi) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\xi) \cdot h_j + r(h)$$

$$= \langle \nabla f(\xi), h \rangle + r(h)$$

If $f : \mathbb{R}^n \to \mathbb{R}^m$, it is differentiable if all coordinate functions $f_1, f_2, \dots f_m$ are differentiable. Then all partial derivatives exist and $L(h) = (\text{Jacobian matrix}) \cdot h$

Theorem: If all partial derivatives exist and are all continuous, then f is differentiable.

! If partial derivatives exist, but are not continuous, then f doesn't need to be differentiable.

Directional Derivatives

Def: Assume $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, $v \in \mathbb{R}^n$ with $\|v\| = 1$. The directional derivative of f at ξ in the direction of v is defined as,

$$D_v f(\xi) = \lim_{t \to 0} \frac{f(\xi + \overset{\in \mathbb{R}}{t} \cdot \overset{\in \mathbb{R}^n,\ \text{direction}}{v}) - f(\xi)}{t}$$

Theorem: $f: \mathbb{R}^n \to \mathbb{R}$ differentiabe at ξ. Then all the directional derivatives exist, and we can compute them as:

$$D_v f(\xi) = (\nabla f(\xi))^T \cdot \underbrace{v}_{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}} = \sum_{i=1}^{n} \overset{\in \mathbb{R}}{v_i} \overset{\text{partial derivative.}}{\frac{\partial f}{\partial x_i}(\xi)}$$

The largest value of all directional derivatives is attained in direction:

$$v = \frac{\nabla f(\xi)}{\|\nabla f(\xi)\|}$$