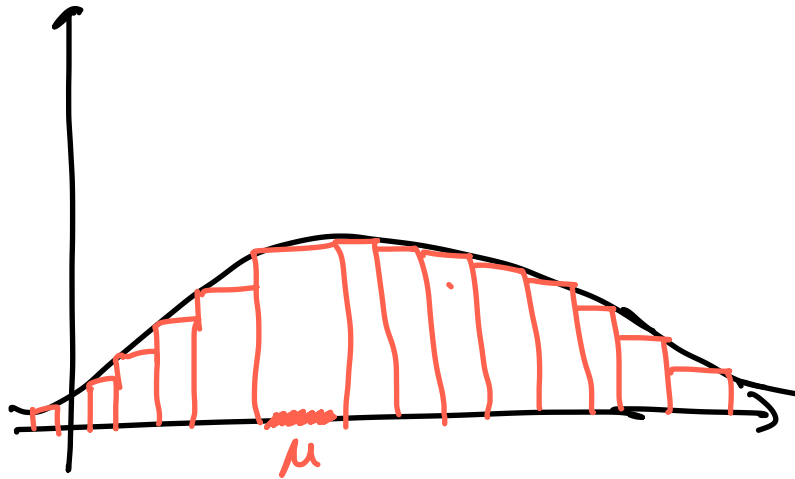


# The Lebesgue Integral on $\mathbb{R}^n$

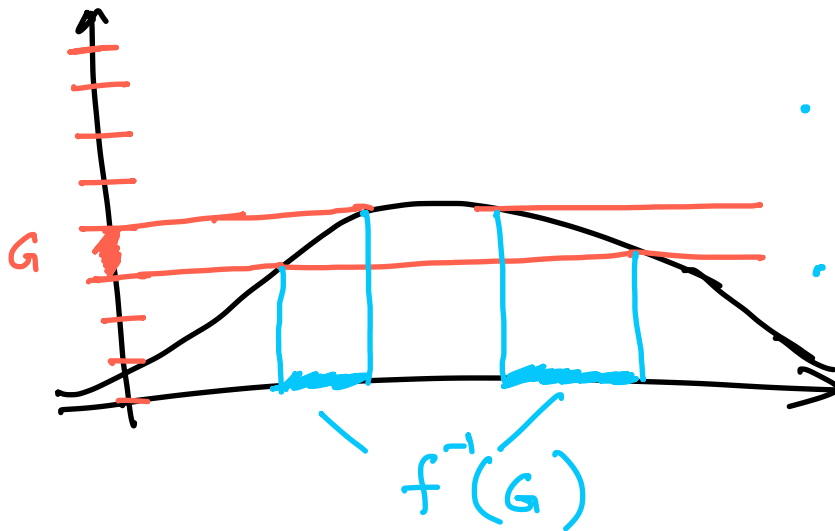
## Intuition:

### Riemann



- bounded
- continuous
- finite set of rectangles.

### Lebesgue

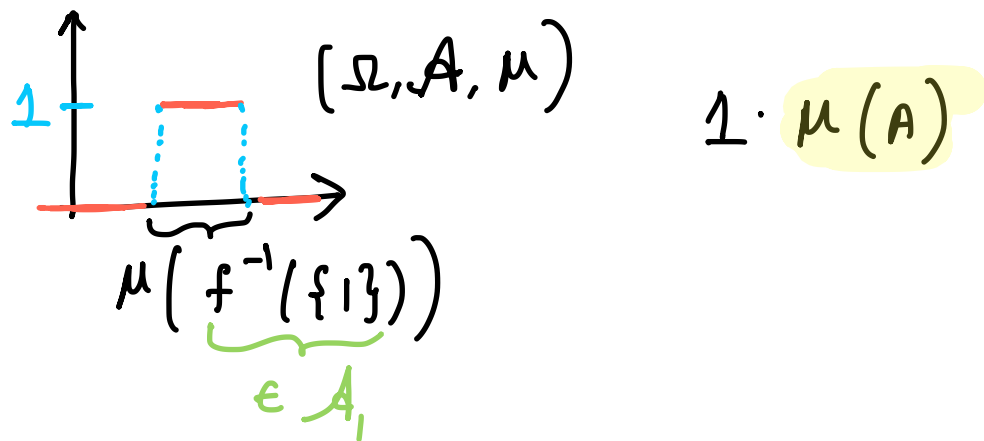


- not bounded
- need not be continuous
- countable sets

Def: A function  $f: (\Omega_1, \mathcal{A}_1) \rightarrow (\Omega_2, \mathcal{A}_2)$  between two measurable spaces is called a measurable if pre-images of measurable sets are measurable:

$$\forall A_2 \in \mathcal{A}_2 : f^{-1}(A_2) \in \mathcal{A}_1$$

$$\hookrightarrow =: \{ \omega \in \Omega_1 \mid f(\omega) \in A_2 \}$$



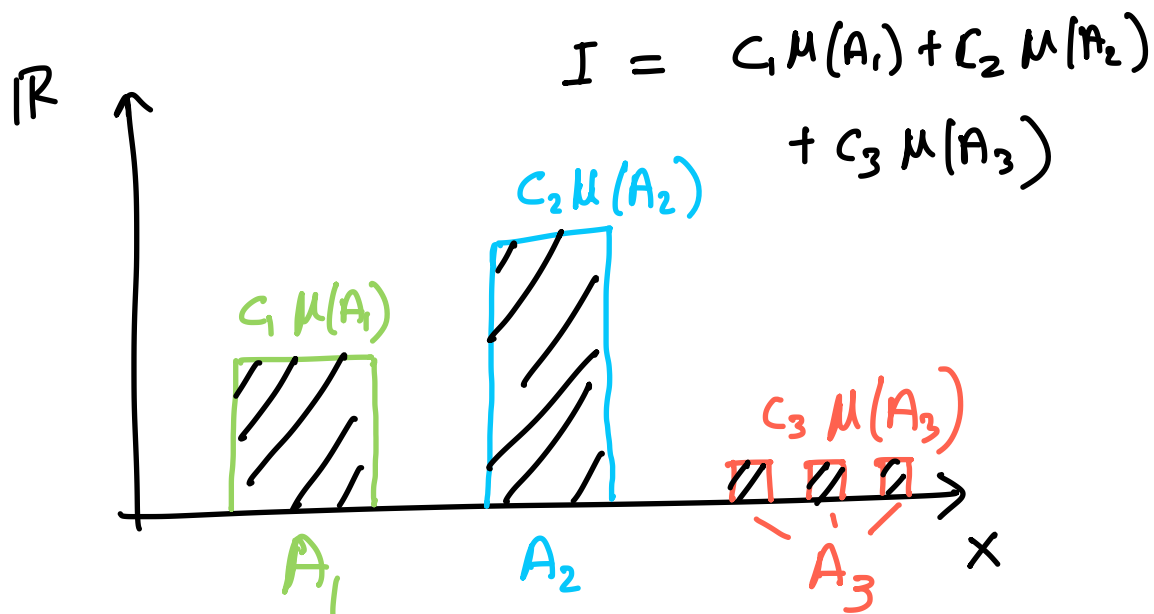
$$(\Omega, \mathcal{A}), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

Characteristic function (also indicator func.)

$$\chi_A: \Omega \rightarrow \mathbb{R}, \chi_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

Def:  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$  is called a simple function if there exist measurable sets  $A_i \subset \mathbb{R}^n$ ,  $c_i \in \mathbb{R}$  such that

$$\phi(x) = \sum_{i=1}^n c_i \chi_{A_i}(x)$$



$$\phi(x) = c_1 \chi_{A_1}(x) + c_2 \chi_{A_2}(x) + c_3 \chi_{A_3}(x)$$

$$I(\phi) = \int \phi d\mu = \sum_{i=1}^n c_i \mu(A_i) \rightarrow \text{Lebesgue integral for simple func.}$$

Problem:  $3 \cdot \infty - 2 \cdot \infty$  ??

For a function  $f^+ : \mathbb{R}^n \rightarrow [0, \infty)$  we define its Lebesgue integral

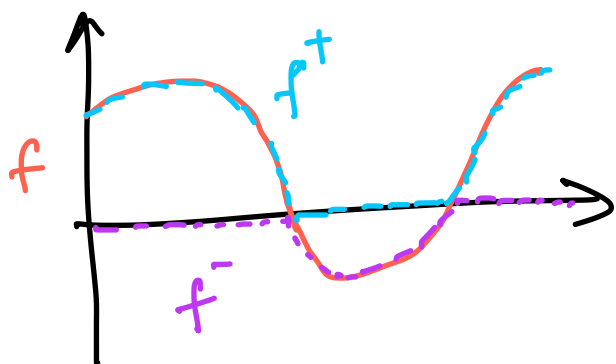
$$\int f^+ d\mu = \sup \left\{ \int \phi d\mu \mid \phi \leq f, \phi \text{ simple} \right\}$$

(might be  $\infty$ )

For a general function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we split the function into positive and negative parts:

$$f = f^+ - f^-, \quad f^+ \geq 0, \quad f^- \geq 0$$

$$\text{where } f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$f = f^+ - f^-$$

Note:  $f^+, f^-$  are measurable if  $f$  is measurable.

If both  $f^+$  and  $f^-$  satisfy  $\int f^+ d\mu < \infty$   
 $\int f^- d\mu < \infty$ , then we call  $f$  integrable  
 and define

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Much more powerful notion than Riemann Integral.

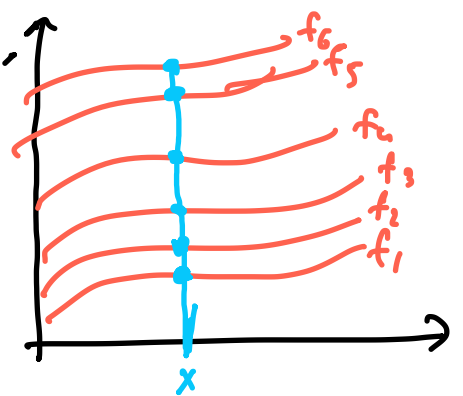
Example:  $\int \chi_{\mathbb{Q}} d\mu = 1 \cdot \mu(\mathbb{Q}) = 0$

## Two important Theorems

Theorem (monotone convergence): Consider a  
 sequence of functions  $f_n : \mathbb{R}^n \rightarrow [0, \infty)$  that is  
 pointwise non-decreasing:  $\forall x \in \mathbb{R}^n, f_{k+1}(x) \geq f_k(x)$ .

Assume that all  $f_k$  are measurable,  
 and that the pointwise limit exists

$$\forall x : \lim f_k(x) =: f(x)$$



Then:

$$\int \lim_{k \rightarrow \infty} f_k(x) dx = \lim_{k \rightarrow \infty} \int f_k(x) dx$$

$\Downarrow$

$$\int f(x) dx$$

Theorem (dominated convergence):

$f_k : B \rightarrow \mathbb{R}$ ,  $|f_k(x)| \leq g(x)$  on  $B$ ,  $g(x)$  is integrable. Assume that the pointwise limit exists:  $\forall x \in B$ ,  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ .

Then:

$$\int \lim_{k \rightarrow \infty} f_k(x) dx = \lim_{k \rightarrow \infty} \int f_k(x) dx$$

$\Downarrow$

$$\int f(x) dx$$

# Partial Derivatives on $\mathbb{R}^n$

Consider  $f: \mathbb{R}^n \rightarrow \mathbb{R}$

Def:  $f$  is called partially differentiable with respect to variable  $x_j$  at point  $\xi \in \mathbb{R}^n$  if the function

$$x_j \mapsto g(x_j) := f(\xi_1, \xi_2, \dots, \xi_{j-1}, x_j, \xi_{j+1}, \dots, \xi_n)$$

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

is differentiable at  $\xi_j \in \mathbb{R}$ .

$j$ -th unit vector =  $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow j$   
 $\in \mathbb{R}$

Notation: 
$$\frac{\partial f(\xi)}{\partial x_j} = \lim_{h \rightarrow 0} \frac{f(\xi + e_j \cdot h) - f(\xi)}{h}$$

$$\mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad f(x) = x_1^2 + x_2^2 \cdot x_1$$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$

If all partial derivatives exist, then the vector of all partial derivatives is called the gradient:


$$\text{grad}(f)(\xi) = \nabla f(\xi) = \begin{pmatrix} \frac{\partial f(\xi)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\xi)}{\partial x_n} \end{pmatrix} \in \mathbb{R}^n$$

If  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we decompose  $f$  into its  $m$  component functions  $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ . We

define the Jacobian matrix:

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$\nabla f_1(x)$   
 $\nabla f_m(x)$

 Even if all partial derivatives exist at  $\xi$ , we do not know if  $f$  is continuous at  $\xi$ .

Example:  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$f(x, y) = \begin{cases} \frac{x \cdot y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = y = 0 \end{cases}$$

For  $(x, y) \neq (0, 0)$

$$\nabla f(x, y) = \left( y \cdot \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad x \cdot \frac{x^2 - y^2}{(x^2 + y^2)^2} \right)$$

$\nabla f(0, 0) = 0$  since  $f(x, 0) = 0 \quad \forall x$ ,  $f(0, y) = 0 \quad \forall y$   
but  $f$  is not continuous at  $0$ .



# Total Derivative

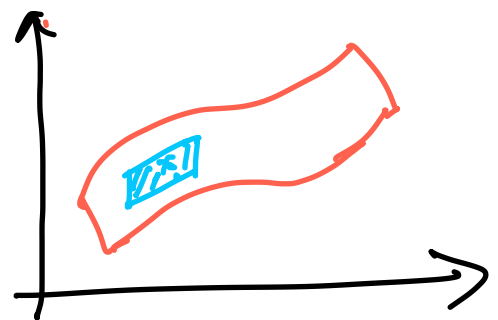
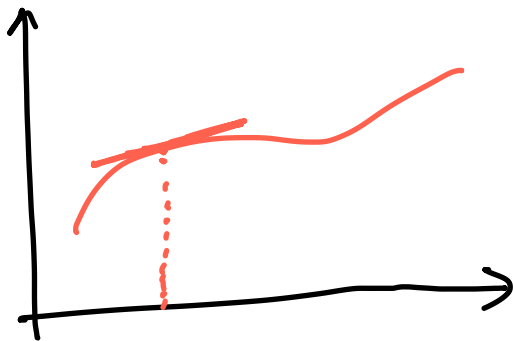
$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad \xi \in U$$

$f$  is differentiable at  $\xi$  if there exists a linear mapping  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

for  $h \in \mathbb{R}^n$ ,

$$f(\xi + h) - f(\xi) = L(h) + r(h)$$

with  $\lim_{h \rightarrow 0} \frac{r(h)}{|h|} \rightarrow 0.$



Intuition:  $f$  is "locally linear"

Theorem:  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable at  $\xi$

- Then  $f$  is continuous at  $\xi$
- The linear functional  $L$  coincides with the gradient:

$$\begin{aligned} f(\xi + h) - f(\xi) &= \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\xi) \cdot h_j + \varepsilon(h) \\ &= \langle \nabla f(\xi), h \rangle + \varepsilon(h) \end{aligned}$$

If  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , it is differentiable if all coordinate functions  $f_1, f_2, \dots, f_m$  are differentiable. Then all partial derivatives exist and  $L(h) = (\text{Jacobian matrix}) \cdot h$

Theorem: If all partial derivatives exist and are all continuous, then  $f$  is differentiable.



If partial derivatives exist, but are not continuous, then  $f$  doesn't need to be differentiable.

# Directional Derivatives

Def. Assume  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable,  $\mathcal{V} \in \mathbb{R}^n$  with  $\|\mathcal{V}\| = 1$ . The directional derivative of  $f$  at  $\xi$  in the direction of  $\mathcal{V}$  is defined as,

$$D_{\mathcal{V}}f(\xi) = \lim_{t \rightarrow 0} \frac{f(\xi + \overset{\substack{\in \mathbb{R} \\ \downarrow}}{t} \cdot \overset{\substack{\in \mathbb{R}^n, \text{ direction} \\ \downarrow}}{\mathcal{V}}) - f(\xi)}{t}$$

Theorem:  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable at  $\xi$ .  
Then all the directional

$\mathcal{V}$

The largest value of all directional derivatives is attained in direction:  $\mathcal{V} = \frac{\nabla f(\xi)}{\|\nabla f(\xi)\|}$