Lebesgue Decomposition

$(X, \mathcal{A}, \lambda)$ measure space

$X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$ (Lebesgue measure) $(\lambda([a,b]) = b-a)$

Another measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$

Def: (a) μ is called absolutely continuous

if $\lambda(A) = 0 \Rightarrow \mu(A) = 0$

for all $A \in \mathcal{B}(\mathbb{R})$. $\mu \ll \lambda$

(b) μ is called singular (w.r.t. λ)

if there is $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$

and $\mu(N^c) = 0$ $\mu \perp \lambda$.

Example: δ_0 Dirac measure $(\delta_0(\{0\}) = 1)$

$N = \{0\}$, $N^c = \mathbb{R} \setminus \{0\}$. $\lambda(N) = 0$.

$\delta_0(\mathbb{R} \setminus \{0\}) = 0$. $\delta_0 \perp \lambda$

Theorem (Decomposition by Lebesgue)

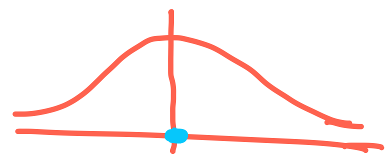
μ, γ prob. measures on $(\Omega, \mathcal{A})$. Then there exists a unique decomposition

$$\gamma = \gamma_{ac} + \gamma_s \quad \text{such that}$$

$$\gamma_{ac} \ll \mu \quad \text{and} \quad \gamma_s \perp \mu .$$

Example: $\gamma = \frac{1}{2}\left(N(0,1), \, \delta_0\right)$

$$\gamma = \gamma_{ac} + \gamma_s$$

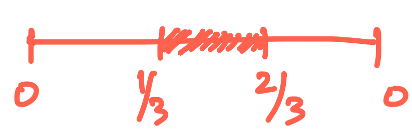


where $\gamma_{ac} = \frac{1}{2} N(0,1)$, $\gamma_s = \frac{1}{2}\delta_0$

Cantor distribution: non-trivial distribution that is singular w.r.t. λ.

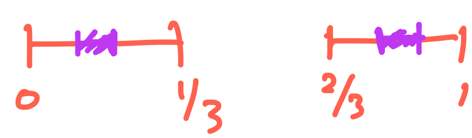
Construct the Cantor set:

- Start with $C_0 := [0,1]$



"remove middle part"

- $C_1 := [0, 1/3] \cup [2/3, 1]$



"remove middle part from all intervals"

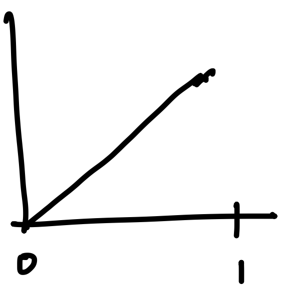
• $C_2 :=$ 

$\vdots$

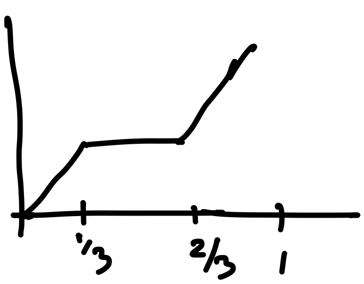
The Cantor set is the limit in this process.

Now construct a prob. distribution:

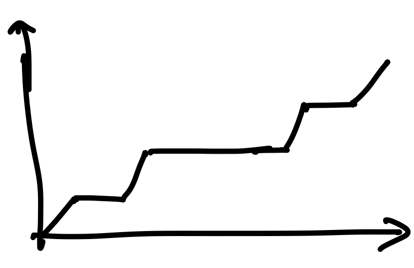
Consider the CDFs of the sets $C_0, C_1, C_2 \ldots\ldots$

C_0: 

uniform on $[0,1]$

C_1: 

uniform on $[0, 1/3] \cup [2/3, 1]$

C_2: 

$\vdots$

Take limit $\to T$. Can prove many interesting properties:

• Cantor set is compact, non-empty, empty interior.

only boundary points.

- The cdf of T is continuous. T is a prob. measure.

- But: $\lambda(C) = 0$

$\Rightarrow \quad \lambda \perp T$

Cumulative Distribution Function

Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Define the function

$$F: \mathbb{R} \to \mathbb{R}, \quad x \mapsto P((-\infty, x])$$

We say that F is a cumulative distribution function (cdf), that satisfies the following properties:

(i) F is monotonically increasing,

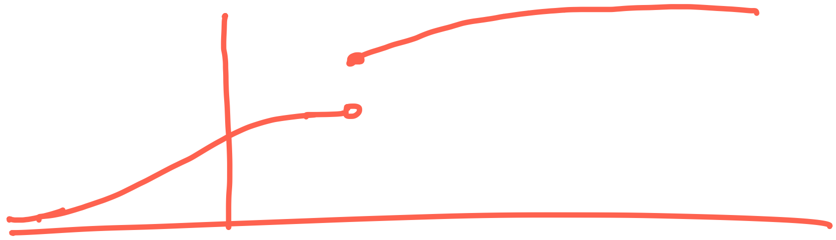
$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1$$

(ii) F is continuous from the right:

$(x_n)_{n \in \mathbb{N}}$ sequence with $x_n \searrow x$

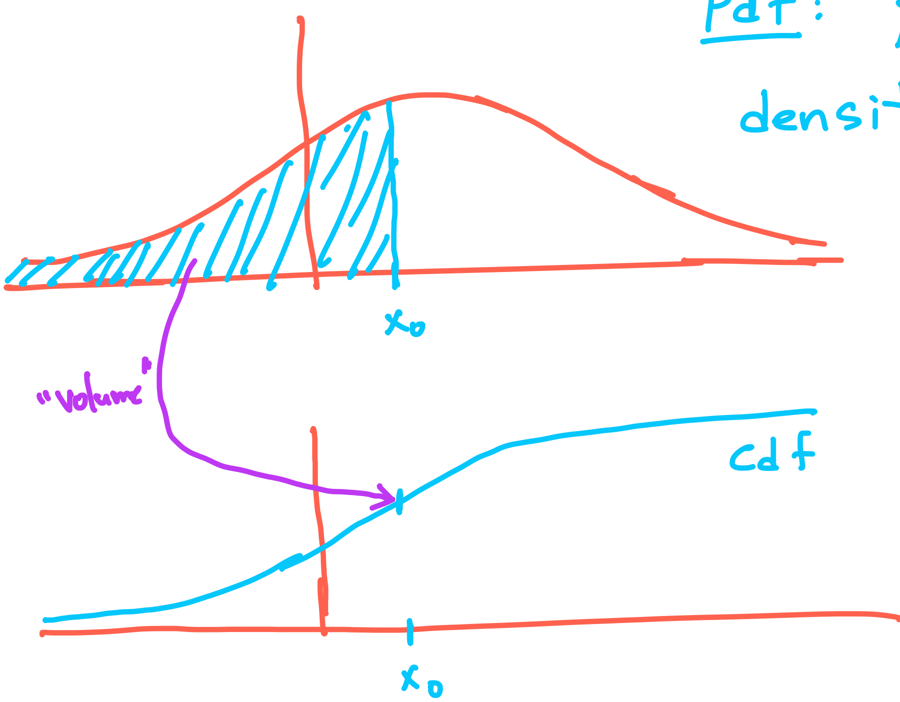
(i.e. $x_n \geq x_{n+1}$ and $x_n \to x$) then also

$$F(x_n) \to F(x)$$



Pdf: prob. density func.

density of normal dist.



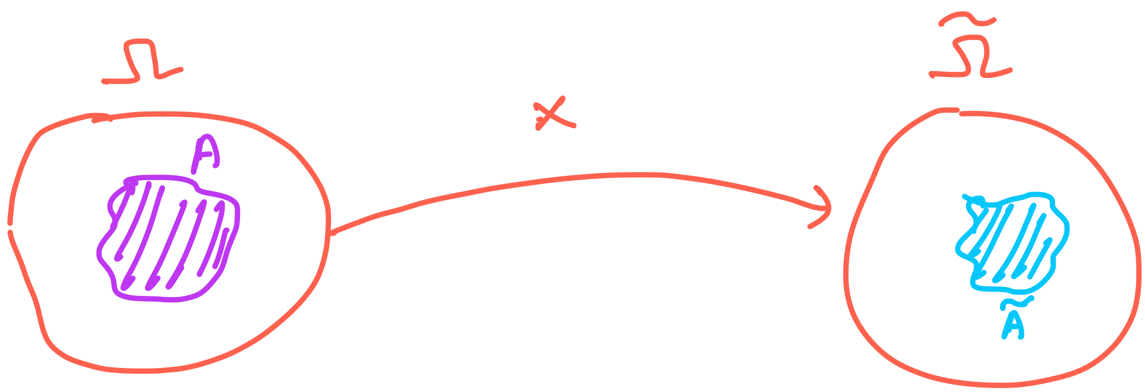
Let $F: \mathbb{R} \to \mathbb{R}$ be a function with properties (i) and (ii). Then there exists a unique probability measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $P\big((-\infty, x]\big) := F(x)$.

Remark: We can go both ways. Given a PDF construct CDF & given a CDF we can construct a unique PDF.

Random Variable

Def: Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\tilde{\Omega}, \tilde{\mathcal{A}})$ be another measurable space. A mapping $X : \Omega \to \tilde{\Omega}$ is called a <u>random variable</u> if X is measurable, i.e.

$$\forall \tilde{A} \in \tilde{\mathcal{A}} : \quad X^{-1}(\tilde{A}) := \{\omega \in \Omega \mid X(\omega) \in \tilde{A}\} \in \mathcal{A}.$$



people in the world

height

A = "people that are at least 6ft"

$\tilde{A}$ = "at least 6ft"

$P(A) = 0.1$

Example: sum of two dice.

$$\Omega = \{ (i,j) \mid i,j \in \{1,2,\dots 6\} \}$$

$$\mathcal{A} = \mathcal{P}(\Omega)$$

$$P(\{(i,j)\}) = \frac{1}{36}$$

$$\tilde{\Omega} = \{2,3,\dots 12\}$$

$$\tilde{\mathcal{A}} = \mathcal{P}(\tilde{\Omega})$$

X "sum of the two dice"

$$X : \Omega \to \{2,3,\dots 12\}, \quad (i,j) \mapsto i+j$$

Is measurable.

Def: A random variable $X : \Omega \to \tilde{\Omega}$ induces a measure on the target space:

For $\tilde{A} \in \tilde{\mathcal{A}}$ we define

$$P_X(\tilde{A}) := P(X^{-1}\{\tilde{A}\})$$

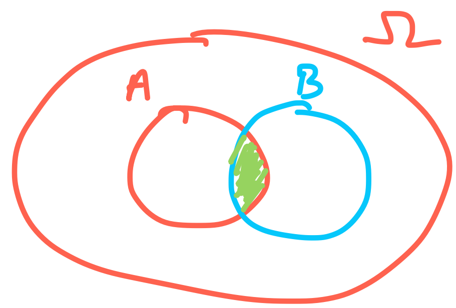
This is a probability measure on $(\tilde{\Omega}, \tilde{\mathcal{A}})$ and it is called the distribution of X.

Def: $X : (\Omega, \mathcal{A}, P) \to (\tilde{\Omega}, \tilde{\mathcal{A}})$. Then the family $\sigma(X) := \{ X^{-1}(\hat{A}) \mid \hat{A} \in \tilde{\mathcal{A}} \}$ is a σ-algebra on Ω and it is called the <u>σ-algebra induced by X.</u>

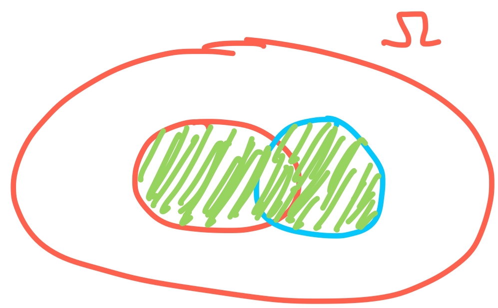
(it is the smallest σ-algebra on Ω that makes X measurable)

Conditional Probabilities

Notation: $P(A \cap B) = P(\text{"A and B"})$



$P(A \cup B) = P(\text{"A or B"})$



Def: Let $(\Omega, \mathcal{A}, P)$ be a probability space.

$A, B \in \mathcal{A}$, $P(B) > 0$. Then

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$ is called

the conditional probability of

A given B.

Theorem: The mapping $P_B : \mathcal{A} \to [0,1]$, $A \mapsto P(A|B)$ is a probability measure on $(\Omega, \mathcal{A})$, it is called the conditional distribution of P with respect to B.

Examples: ① two dice

$$P(\text{"sum is 9"} \mid \text{"first die was 3"})$$

② Ω = all persons on earth

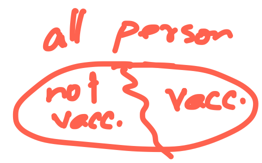
$\mathcal{A} = \mathcal{P}(\Omega)$

P = "uniform"

event A: "person has been vaccinated"

B: "person has disease"

$P(\text{disease} \mid \text{vaccinated}) \to$



$P(\text{vaccinated} \mid \text{disease})$