# Linear Mappings

**Def:** Let $U, V$ be Vector spaces over the same field $F$. A mapping $T: U \to V$ is called a linear map if $\forall\, u_1, u_2 \in U, \lambda \in F$

$$t(u_1 + u_2) = t(u_1) + t(u_2)$$

$$t(\lambda u_1) = \lambda t(u_1)$$

The set of all linear mappings from $U \to V$ is denoted $\mathcal{L}(U, V)$.

If $U = V$, then we denote $\mathcal{L}(U)$.

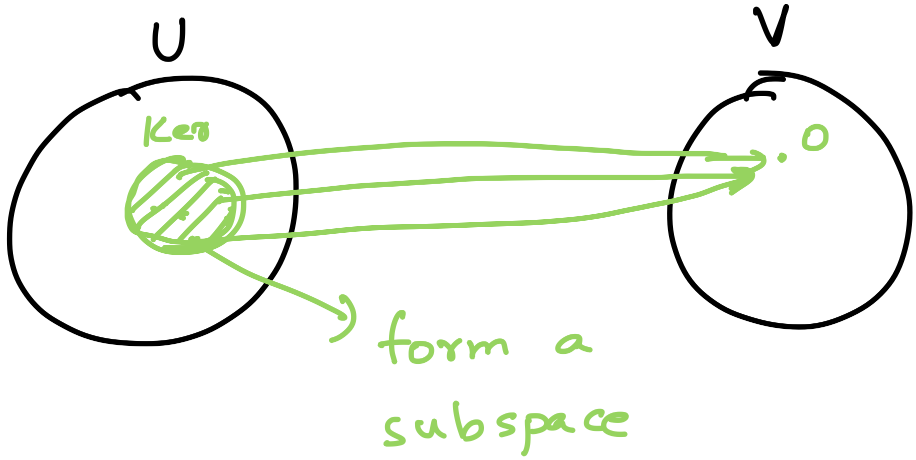
**Examples:** • $T: \mathbb{C}([a,b]) \to \mathbb{R}$, $t \mapsto \int_a^b t(x)\,dx$ (integration)

• $D: \mathbb{C}^\infty([a,b]) \to \mathbb{C}^\infty([a,b])$, $t \mapsto t'$ (differentiation)

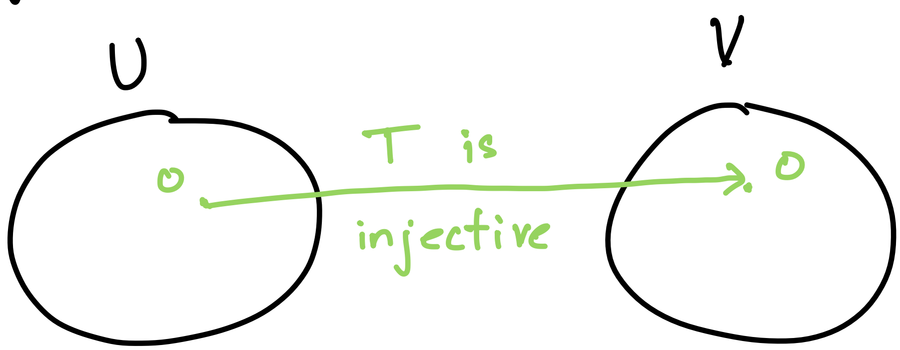
**Def:** $T \in \mathcal{L}(U, V)$. Then Kernel of $T$ (null-space of $T$) is defined as.

$$\ker(T) := \operatorname{null}(T) := \{u \in U \mid Tu = 0\}$$

**Proposition:** • $\ker(T)$ is a subspace of $U$.




• $T$ injective iff $\ker(T) = \{0\}$




**Def:** The range of $T$ (image of $T$) is defined as,

$$\text{range}(T) := \text{image}(T) := \{Tu \mid u \in U\}$$

**Proposition:** • The range is always a subspace of $V$.

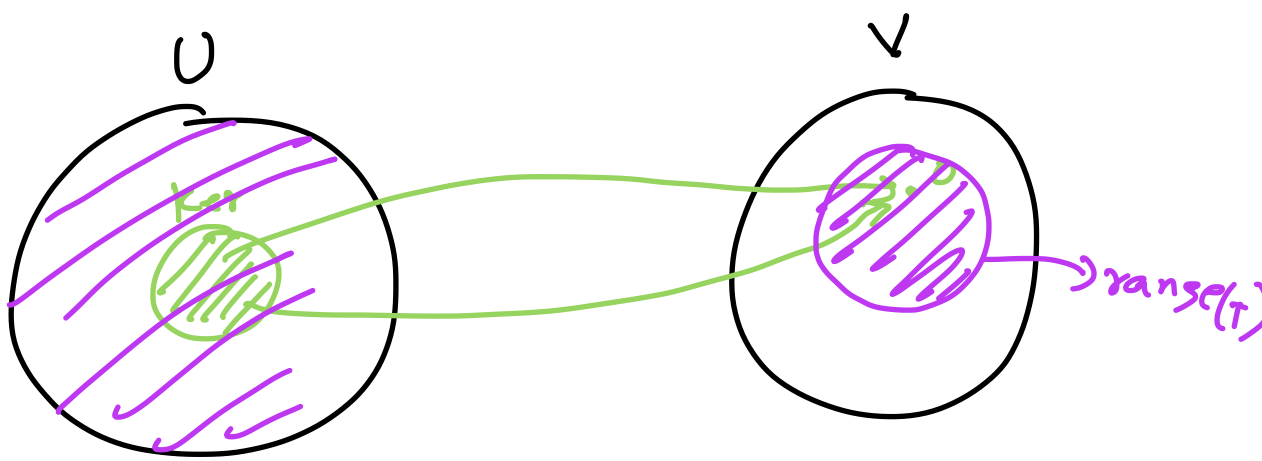
• $T$ is surjective iff $\text{rang}(T) = V$

**Def:** Let $V'$ be any subset of $V$ i.e. $V' \subset V$.

The pre-image of $V'$ is defined as

$$T^{-1}(V') := \{ u \in U \mid Tu \in V' \}$$

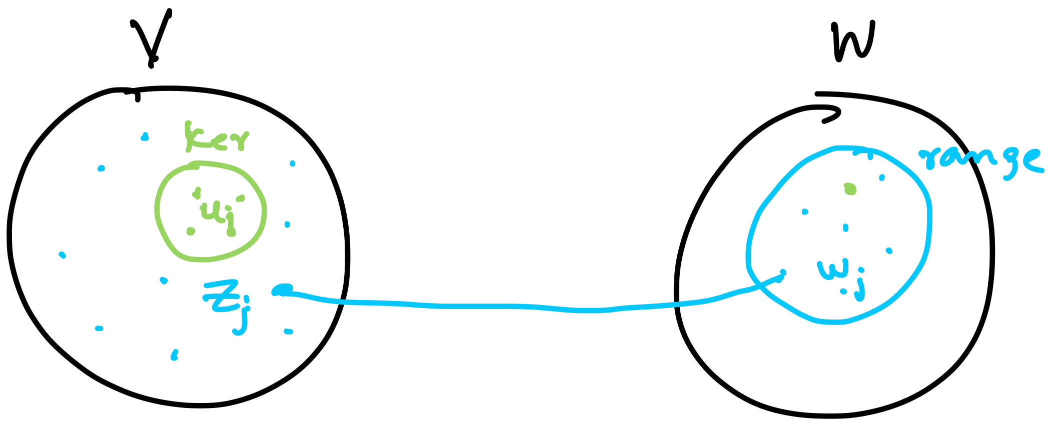
**Proposition:** If $V' \subset V$ is a subspace of $V$, then $T^{-1}(V')$ is a subspace of $U$.




**Theorem:** Let $V$ be finite-dim, $W$ any VS, $T \in \mathcal{L}(V, W)$. Let $(u_1, \dots, u_n)$ be a basis of $\ker(T) \subset V$. Let $w_1 \dots w_m$ be a basis of $\operatorname{range}(T) \subset W$. Then. $u_1 \dots u_n, T^{-1}(w_1) \dots T^{-1}(w_m) \subset V$ form a basis of $V$.

In particular, $\dim(V) = \dim(\ker(T)) + \dim(\operatorname{range}(T))$.

**Proof:** Denote $T^{-1}(w_1) = z_1, \dots, T^{-1}(w_m) = z_m$




$$\boxed{\text{Step 1: } V \subset \operatorname{span}\{u_1 \dots u_n, z_1 \dots z_m\}}$$

Let $v \in V$ consider $Tv \in \text{range}(T)$

$\Rightarrow \exists\ \lambda_1 \dots \lambda_m$ s.t.

$$Tv = \lambda_1 w_1 + \lambda_2 w_2 + \dots \lambda_m w_m$$

$$= \lambda_1 T(z_1) + \lambda_2 T(z_2) \dots \lambda_m T(z_m)$$

$$= T(\lambda_1 z_1 + \lambda_2 z_2 + \dots \lambda_m z_m)$$

$$\Rightarrow Tv - T(\lambda_1 z_1 + \lambda_2 z_2 + \dots \lambda_m z_m) = 0$$

$$\Rightarrow T(\underbrace{v - (\lambda_1 z_1 + \lambda_2 z_2 + \dots \lambda_m z_m)}_{\in \ker(T)}) = 0$$

$\Rightarrow \exists\, \mu_1 \dots \mu_n$ s.t.

Reminder: $u_1, \dots, u_n$ are basis of $\ker(T)$

$$v - (\lambda_1 z_1 + \lambda_2 z_2 + \dots \lambda_m z_m) = \mu_1 u_1 + \mu_2 u_2 + \dots \mu_n u_n$$

$$\Rightarrow \quad v = \lambda_1 z_1 + \lambda_2 z_2 \dots + \lambda_m z_m + \mu_1 u_1 + \mu_2 u_2 + \dots \mu_n u_n$$

**Step 2:** $u_1, u_2 \dots u_n, z_1, z_2 \dots z_m$ are linearly independent.

Assume that $\mu_1 u_1 + \dots \mu_n u_n + \lambda_1 z_1 + \dots \lambda_m z_m = 0$ (*)

Now consider: $\lambda_1 w_1 + \dots \lambda_m w_m$

$$= \lambda_1 T(z_1) + \dots + \lambda_m T(z_m)$$

$$= \lambda_1 T(z_1) + \dots + \lambda_m T(z_m) + \underbrace{\mu_1 T(u_1) + \dots + \mu_n T(u_n)}_{0}$$

$$= T(\underbrace{\lambda_1 z_1 + \dots \lambda_m z_m + \mu_1 u_1 + \dots + \mu_n u_n}_{=0 \text{ by } (*)}) = 0$$

$$\Rightarrow \lambda_1 w_1 + \dots \lambda_m w_m = 0 \xRightarrow[\text{basis}]{w_1 \dots w_m} \lambda_1 = 0 \dots \lambda_m = 0$$

$$\Rightarrow \mu_1 u_1 + \dots \mu_n u_n = 0$$

$\Rightarrow \mu_1 = \mu_2 = \dots = \mu_n = 0$ since $u_1 \dots u_n$ are basis $\blacksquare$

**Proposition:** $T \in \mathcal{L}(V,V)$, $V$ is finite-dim.
Then the following statements are equivalent.

(i) $T$ is injective

(ii) $T$ is surjective

(iii) $T$ is bijective

**Proof:** Direct consequence of theorem.

⚠ Does not hold in $\infty$-dim spaces.

## Matrices and Linear Maps

**Notation:**

$$A = m\text{-rows}\left\{ \overbrace{\begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \vdots & & \\ \vdots & & \\ a_{m1} & \cdots\cdots & a_{mn} \end{pmatrix}}^{n\text{-cols}} \right. = (a_{ij})_{\substack{i=1,\dots m \\ j=1,\dots n}}$$

Consider $T \in \mathcal{L}(V, W)$, $V, W$ finite-dim,

let $v_1 \dots v_n$ be a basis of $V$

$w_1 \dots w_m$ be a basis of $W$

- $v = \lambda_1 v_1 \dots + \lambda_n v_n$

$$T(v) = T(\lambda_1 v_1 + \dots + \lambda_n v_n)$$
$$= \lambda_1 T(v_1) + \dots + \lambda_n T(v_n)$$

- Each image vector $T(v_j)$ can be expressed in basis $w_1 \dots w_m$:

there exist co-efficients $a_{1j}, \dots a_{mj}$ s.t.

$$\boxed{T(v_j) = a_{1j} w_1 + \dots + a_{mj} w_m}$$

- We now stack these co-efficients in a matrix:

m rows, one for each basis vector of W.

$$\begin{pmatrix} a_{11} & \dots & \boxed{a_{1j}} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} = $$

matrix of mapping $T$ w.r.t. the basis $v_1 \dots v_n$ of $V$, $w_1 \dots w_m$ of $W$

col $j$

n-cols, one for each basis vector of V.

**Notation:** Let $T: V \to W$ be linear, let $\mathcal{B}$ a basis of $V$, $\mathcal{C}$ basis of $W$. We denote by $M(T, \mathcal{B}, \mathcal{C})$ the matrix corresponding to $T$ w.r.t. bases $\mathcal{B}$ and $\mathcal{C}$.

**Convenient properties of matrices:** Let $V, W$ be vector spaces, consider the bases fixed. Let $S, T \in \mathcal{L}(V, W)$,

- $M(S+T) = M(S) + M(T)$
- $M(\lambda S) = \lambda M(S)$

linear properties of mapping extend to matrices.

- For $v = \lambda_1 v_1 + \dots \lambda_n v_n$ we have that

$$\underbrace{T(v)}_{\text{image of } v \text{ under } T} = \underbrace{M(T)\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}}_{\text{matrix-vector product}}$$

where $(v_1, \dots v_n)$ is basis of $V$

- $T: U \to V$, $S: V \to W$ linear, then

$$M(\underbrace{S \circ T}_{\text{composition of the maps } S \text{ and } T.}) = M(S) \cdot M(T)$$

# Invertible maps and Matrices

Def: $T \in \mathcal{L}(V,W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W,V)$ such that

$$S \circ T = Id_V \quad \text{and} \quad T \circ S = Id_W$$

The map $S$ is called the inverse of $T$, denoted by $T^{-1}$

Remark: Inverse maps exist and are unique.

Prop: A linear map is invertible iff it is injective and surjective i.e. bijective.

Proof: " $\Rightarrow$ " invertible $\Rightarrow$ injective:

suppose $T(u) = T(v)$. Then $u = T^{-1}(T(u))$

$= T^{-1}(T(v)) = v \Rightarrow u = v \Rightarrow$ injective

invertible $\Rightarrow$ surjective: $w \in W$. Then

$w = T(T^{-1}(w)) \Rightarrow w \in$ range of $T$

$\Rightarrow$ surjective

" $\Leftarrow$ " injective & surjective $\Rightarrow$ invertible

Let $w \in W$. There exists unique $v \in V$. s.t. $T(u) = w$

Define the mapping: $S(w) = v$. Clearly have $T \circ S = Id$

Let $v \in V$. Then $T((S \circ T) v)$

$= (T \circ S)(Tv) = Id \circ Tv = Tv$

↳ composition of mappings is associative

$\Rightarrow (S \circ T) v = v \Rightarrow S \circ T = Id \Rightarrow S$ is inverse of $T$

Still need to show $S$ is a linear mapping

Let $y_1, y_2 \in W, \alpha \in F$ : $S(y_1 + y_2) = S(y_1) + S(y_2)$

$S(\alpha y_1) = \alpha S(y_1)$

Let $x_1, x_2 \in V$. s.t. $T(x_i) = y_i$. Then $S(y_i) = x_i$

$$
\begin{aligned}
S(y_1 + y_2) &= S(T(x_1) + T(x_2)) \\
&= S(T(x_1 + x_2)) \\
&= x_1 + x_2 \\
&= S(y_1) + S(y_2)
\end{aligned}
$$

$$
\begin{aligned}
S(\alpha y_1) &= S(\alpha T(x_1)) \\
&= S(T(\alpha x_1)) \\
&= \alpha x_1 \\
&= \alpha S(y_1)
\end{aligned}
$$

$\Rightarrow S$ is a linear transform.

# Inverse Matrix

**Def:** A square matrix $A \in F^{n \times n}$ is invertible if there exists a square matrix $B \in F^{n \times n}$ such that: $A \cdot B = B \cdot A = Id = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

The matrix $B$ is called the inverse matrix, and is denoteb by $A^{-1}$.

**Proposition:** The inverse matrix represents the inverse of the corresponding linear map, that is: $T: V \to V$

$$\underbrace{M(T^{-1})}_{\text{matrix of inverse map}} = \underbrace{(M(T))^{-1}}_{\text{inverse matrix of the original map}}$$

In particular, a matrix is invertible iff the corresponding map is invertible.

## Remarks:

- The inverse matrix does not always exist
- $(A^{-1})^{-1} = A$, $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- $A^t$ invertible $\Longleftrightarrow$ $A$ invertible

$$(A^t)^{-1} = (A^{-1})^t$$

- $A \in F^{n \times n}$ invertible $\Longleftrightarrow$ $\text{rank}(A) = n$
- The set of all invertible matrices is called general linear group:

$$GL(n, F) = \{ A \in F^{n \times n} \mid A \text{ invertible} \}$$