

## Linear Mappings

Def: Let  $U, V$  be vector spaces over the same field  $F$ . A mapping  $T: U \rightarrow V$  is called a linear map if  $\forall u_1, u_2 \in U, \lambda \in F$

$$T(u_1 + u_2) = T(u_1) + T(u_2)$$

$$T(\lambda u_1) = \lambda T(u_1)$$

The set of all linear mappings from  $U \rightarrow V$  is denoted  $\mathcal{L}(U, V)$ .

If  $U = V$ , then we denote  $\mathcal{L}(U)$ .

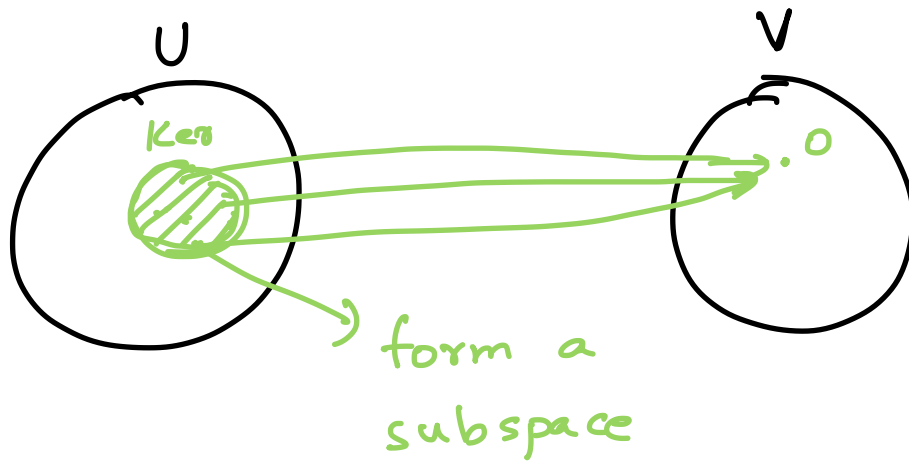
Examples:

- $\cdot T: \mathcal{C}([a, b]) \rightarrow \mathbb{R}, t \mapsto \int_a^b t(x) dx$   
(integration)
- $\cdot D: \mathcal{C}^\infty([a, b]) \rightarrow \mathcal{C}^\infty([a, b]), t \mapsto t'$  (differentiation)

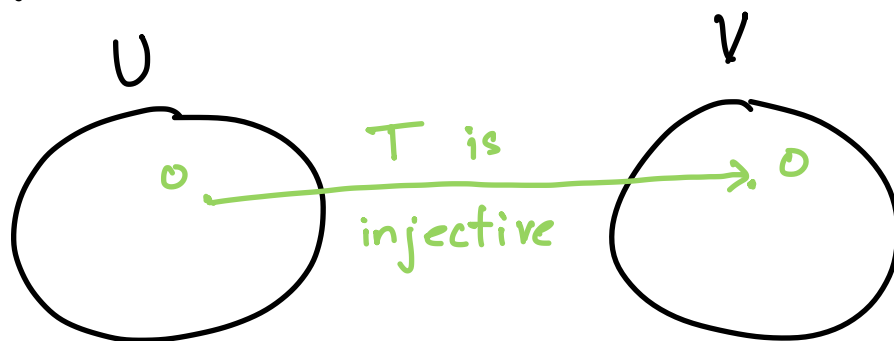
Def:  $T \in \mathcal{L}(U, V)$ . Then kernel of  $T$  (null-space of  $T$ ) is defined as.

$$\text{ker}(T) := \text{null}(T) := \{u \in U \mid Tu = 0\}$$

Proposition:  $\cdot \text{Ker}(T)$  is a subspace of  $U$ .



$\cdot T$  injective iff  $\text{Ker}(T) = \{0\}$



Def: The range of  $T$  (image of  $T$ ) is defined as,

$$\text{range}(T) := \text{image}(T) := \{Tu \mid u \in U\}$$

Proposition:  $\cdot$  The range is always a subspace of  $V$ .

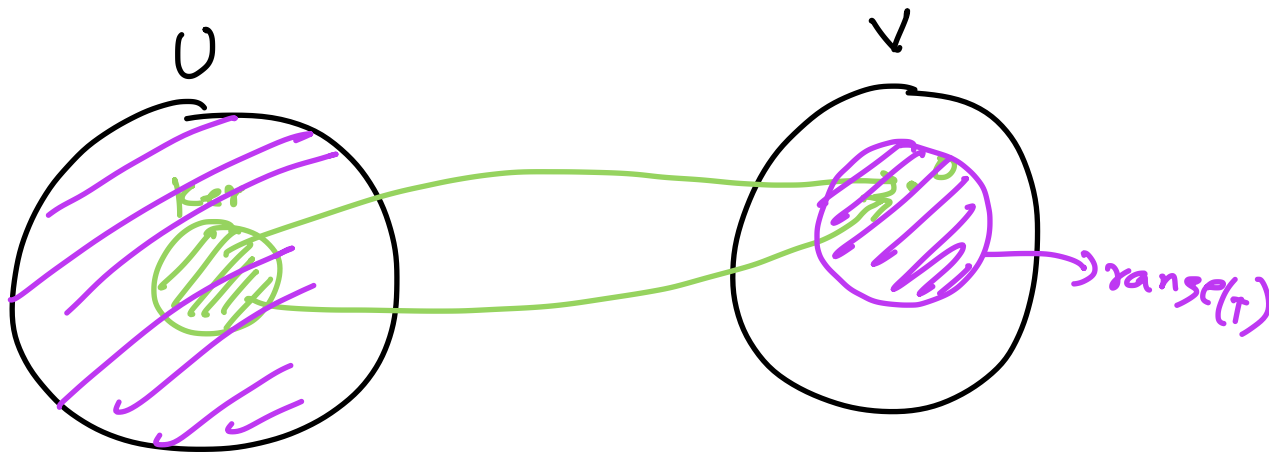
$\cdot T$  is surjective iff  $\text{rang}(T) = V$

Def: Let  $v'$  be any subset of  $V$  i.e.  $v' \subset V$ .

The pre-image of  $v'$  is defined as

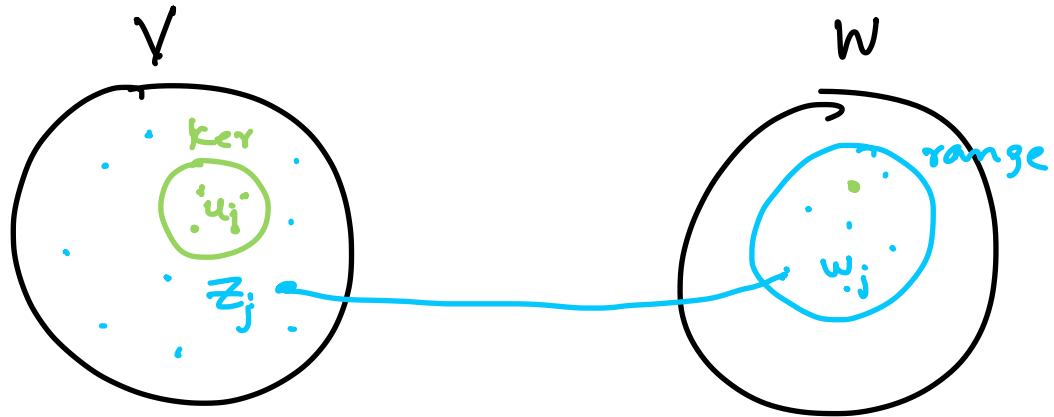
$$T^{-1}(v') := \{u \in U \mid Tu \in v'\}$$

Proposition: If  $v' \subset V$  is a subspace of  $V$ , then  $T^{-1}(v')$  is a subspace of  $U$ .



Theorem: Let  $V$  be finite-dim,  $W$  any VS,  
 $T \in \mathcal{L}(V, W)$ . Let  $(u_1, \dots, u_n)$  be a basis of  $\ker(T) \subset V$ .  
Let  $w_1, \dots, w_m$  be a basis of  $\text{range}(T) \subset W$ . Then,  
 $u_1, \dots, u_n, T^{-1}(w_1), \dots, T^{-1}(w_m) \subset V$  form a basis of  $V$ .  
In particular,  $\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$ .

Proof: Denote  $T^{-1}(w_1) = z_1, \dots, T^{-1}(w_m) = z_m$



Step 1:  $V \subset \text{span} \{u_1, \dots, u_n, z_1, \dots, z_m\}$

Let  $v \in V$  consider  $Tv \in \text{range}(T)$

$\Rightarrow \exists \lambda_1, \dots, \lambda_m$  s.t.

$$Tv = \lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_m w_m$$

$$= \lambda_1 T(z_1) + \lambda_2 T(z_2) + \dots + \lambda_m T(z_m)$$

$$= T(\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m)$$

$$\Rightarrow Tv - T(\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m) = 0$$

$$\Rightarrow T\left(\underbrace{v - (\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m)}_{\in \text{ker}(T)}\right) = 0$$

$\Rightarrow \exists \mu_1 \dots \mu_n$  s.t.

Reminder:  $v_1 \dots v_n$  are basis of  $\ker(T)$

$$v - (\lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m) = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_n v_n$$

$$\Rightarrow v = \lambda_1 z_1 + \lambda_2 z_2 + \dots + \lambda_m z_m + \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_n v_n$$

Step 2:  $u_1, u_2, \dots, u_n, z_1, z_2, \dots, z_m$  are linearly independent.

Assume that  $\mu_1 u_1 + \dots + \mu_n u_n + \lambda_1 z_1 + \dots + \lambda_m z_m = 0$  \*

Now consider:  $\lambda_1 w_1 + \dots + \lambda_m w_m = \lambda_1 T(z_1)$

Proposition:  $T \in \mathcal{L}(V, V)$ ,  $V$  is finite-dim.

Then the following statements are equivalent.

- (i)  $T$  is injective
- (ii)  $T$  is surjective
- (iii)  $T$  is bijective

Proof: Direct consequence of theorem.

⚠ Does not hold in  $\infty$ -dim spaces.

## Matrices and Linear Maps

Notation:

$$A = \begin{matrix} & \underbrace{\hspace{10em}}_{n \text{ - cols}} \\ \underbrace{\hspace{1em}}_{m \text{ - rows}} \left( \begin{array}{cccc} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{array} \right) & = & (a_{ij})_{\substack{i=1, \dots, m \\ j=1, \dots, n}} \end{matrix}$$

Consider  $T \in \mathcal{L}(V, W)$ ,  $V, W$  finite-dim,

let  $v_1, \dots, v_n$  be a basis of  $V$

$w_1, \dots, w_m$  be a basis of  $W$

•  $v = \lambda_1 v_1 + \dots + \lambda_n v_n$

$$T(v) = T(\lambda_1 v_1 + \dots + \lambda_n v_n)$$

$$= \lambda_1 T(v_1) + \dots + \lambda_n T(v_n)$$

• Each image vector  $T(v_j)$  can be expressed in basis  $w_1, \dots, w_m$ :

there exist co-efficients  $a_{1j}, \dots, a_{mj}$  s.t.

$$T(v_j) = a_{1j} w_1 + \dots + a_{mj} w_m$$

• We now stack these co-efficients in a matrix:

$m$  rows,  
one for each  
basis vector  
of  $W$ .

$$\begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

matrix of  
mapping  $T$   
w.r.t. the basis  
 $v_1, \dots, v_n$  of  $V$   
 $w_1, \dots, w_m$  of  $W$

$n$ -cols, one for each basis vector of  $V$ .

Notation: Let  $T: V \rightarrow W$  be linear, let  $B$  a basis of  $V$ ,  $\mathcal{C}$  basis of  $W$ . We denote by  $M(T, B, \mathcal{C})$  the matrix corresponding to  $T$  w.r.t. bases  $B$  and  $\mathcal{C}$ .

Convenient properties of matrices: Let  $V, W$  be vector spaces, consider the bases fixed. Let  $S, T \in \mathcal{L}(V, W)$ ,

- $M(S+T) = M(S) + M(T)$
- $M(\lambda S) = \lambda M(S)$

} linear properties of mapping extend to matrices.

• For  $v = \lambda_1 v_1 + \dots + \lambda_n v_n$  we have that

$$\underbrace{T(v)}_{\text{image of } v \text{ under } T} = M(T) \underbrace{\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}}_{\text{matrix-vector product}} \text{ where } (v_1, \dots, v_n) \text{ is basis of } V$$

•  $T: U \rightarrow V$ ,  $S: V \rightarrow W$  linear, then

$$M(\underbrace{S \circ T}) = M(S) \cdot M(T)$$

↳ composition of the maps  $S$  and  $T$ .



## Invertible maps and Matrices

Def:  $T \in \mathcal{L}(V, W)$  is called invertible if there exists a linear map  $S \in \mathcal{L}(W, V)$  such that

$$S \circ T = \text{Id}_V \quad \text{and} \quad T \circ S = \text{Id}_W$$

The map  $S$  is called the inverse of  $T$ , denoted by  $T^{-1}$

Remark: Inverse maps exist and are unique.



# Inverse Matrix

Def: A square matrix  $A \in F^{n \times n}$  is invertible if there exists a square matrix  $B \in F^{n \times n}$  such that:  $A \cdot B = B \cdot A = \text{Id} = \begin{pmatrix} 1 & & 0 \\ & \dots & \\ 0 & & 1 \end{pmatrix}$

The matrix  $B$  is called the inverse matrix, and is denoted by  $A^{-1}$ .

Proposition: The inverse matrix represents the inverse of the corresponding linear map, that is:  $T: V \rightarrow V$

$$\underbrace{M(T^{-1})}_{\text{matrix of inverse map}} = \underbrace{(M(T))^{-1}}_{\text{inverse matrix of the original map}}$$

In particular, a matrix is invertible iff the corresponding map is invertible.

## Remarks:

• The inverse matrix does not always exist

•  $(A^{-1})^{-1} = A$ ,  $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

•  $A^t$  invertible  $\Leftrightarrow A$  invertible

$$(A^t)^{-1} = (A^{-1})^t$$

•  $A \in F^{n \times n}$  invertible  $\Leftrightarrow \text{rank}(A) = n$

• The set of all invertible matrices is called general linear group:

$$GL(n, F) = \{ A \in F^{n \times n} \mid A \text{ invertible} \}$$