Product space, joint distribution

Consider two measurable spaces $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$.

Define the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with

$$\Omega_1 \times \Omega_2 = \{ (\omega_1, \omega_2) \mid \omega_1 \in \Omega_1, \ \omega_2 \in \Omega_2 \}$$

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \{ A_1 \times A_2 \mid A_1 \in \mathcal{A}_1, \ A_2 \in \mathcal{A}_2 \}.$$

Consider two RVs. $X_1 : (\Omega, \mathcal{A}, P) \to (\Omega_1, \mathcal{A}_1)$

$X_2 : (\Omega, \mathcal{A}, P) \to (\Omega_2, \mathcal{A}_2)$

$$X := (X_1, X_2) : (\Omega, \mathcal{A}, P) \to (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$$

$$(X_1, X_2)(\omega) = (X_1(\omega), X_2(\omega))$$

The distribution $P_{(X_1, X_2)}$ on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is called the joint distribution of X_1 and X_2.

Example in ML: (X, Y) where X is the input data, Y is the label.

Product measure: Let $(\Omega_1, \mathcal{A}_1, P_1)$, $(\Omega_2, \mathcal{A}_2, P_2)$ be two probability spaces. We define the product measure $P_1 \otimes P_2$ on the product space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ as

$$(P_1 \otimes P_2)(A_1 \times A_2) := P_1(A_1) \cdot P_2(A_2)$$

Theorem: Two RVs X_1, X_2 are independent if and only if their joint distribution coincides with the product distributions:

$$P_{(X_1, X_2)} = P_1 \otimes P_2$$

Marginal Distributions

Consider the joint distribution $P_{(X_1, X_2)}$ of two RVs. $X := (X_1, X_2)$. The <u>marginal distribution of X w.r.t. X_1</u> is the original distribution of X_1 on $(\Omega_1, \mathcal{A}_1)$, namely P_{X_1}. Similarly for X_2 as well.

Example in the discrete case:

$Y \backslash X$	x_1	x_2	x_3	Σ
y_1	p_{11}	p_{12}	p_{13}	$p_{11}+p_{12}+p_{13} = P(Y=y_1)$
y_2	p_{21}	p_{22}	p_{23}	$p_{21}+p_{22}+p_{23} = P(Y=y_2)$
Σ	$p_{11}+p_{21} = P(X=x_1)$	$p_{12}+p_{22} = P(X=x_2)$	$p_{13}+p_{23} = P(X=x_3)$	

marginal dist. w.r.t. Y.

Marginal distributions in case of densities

$X, Y : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. $Z := (X, Y)$.

Assum that the joint distribution of Z has a desity f on $\mathbb{R}^2$. Then we have the following statements:

(1) Both X and Y have densities on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by.

joint dist.

sum over y

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

joint dist.

sum over x

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

(2) X and Y are independent iff

$$\underbrace{f(x,y)}_{\text{joint}} = \underbrace{f_X(x)}\cdot \underbrace{f_Y(y)}_{\text{marginals.}} \quad \text{a.s.}$$

almost surely.

Mixed cases

For example, consider X is a continuous RV. with density and Y a discrete RV.

Say, X = image (2d - continuous signal)

Y = "cat", "dog" .... discrete

Special case: marginals of multivariate Normal.

2 dim Consider a 2-dim normal RV $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with mean $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^2$ and cov. $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$

Then the marginal dist. of X w.r.t. x_1 is again a normal distribution with mean μ_1 and var σ_1^2.

sum up the y-direction

↳ marginal w.r.t. x is normal

<u>n-dim</u>: $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$. Group the

Variables $S = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \mathbb{R}^k$, $T = \begin{pmatrix} x_{k+1} \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{n-k}$

Want to look at the marginal of X w.r.t. S. $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$ mean, $\mu_S := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix}$, $\mu_T := \begin{pmatrix} \mu_{k+1} \\ \vdots \\ \mu_n \end{pmatrix}$

$$\Sigma = \begin{array}{c} \left(\begin{array}{c|c} \Sigma_{SS} & \Sigma_{ST} \\ \hline \Sigma_{TS} & \Sigma_{TT} \end{array} \right) \begin{array}{l} \}\,k \\ \\ \end{array} \\ \underbrace{\qquad\quad}_{k} \qquad\qquad \end{array} \in \mathbb{R}^{n \times n}$$

Now the marginal of X w.r.t. S is a normal dist. on $\mathbb{R}^k$ with mean μ_S and cov Σ_{SS}.

In summary: Marginals of normal dist. are again normal dist.

Conditional Distributions

Discrete case:

Know conditional probabilities: $P(A|B)$ defined for events $A, B \in \mathcal{A}$, and $P(B) > 0$.

Let $X, Y : (\Omega, \mathcal{A}, P) \to \mathbb{R}$ be discrete RV, $y \in \mathbb{R}$ such that $P(Y=y) > 0$. Then we can define the conditional probability measure

$$P_{X|Y=y} : A \longmapsto P(X \in A \mid Y=y).$$

This is a probability measure.

For general RV this is very complicated!

(skipping)

Conditional distributions in case of densities:

Assume $Z := (X, Y)$ has a joint density $f : \mathbb{R}^2 \to \mathbb{R}$, and marginal densities $f_X, f_Y : \mathbb{R} \to \mathbb{R}$.

Then the function,

$$f_{X|Y=y}(x) := \frac{f(x,y)}{f_Y(y)}$$

is then also a density on $\mathbb{R}$, called the conditional density of X given $Y=y$.

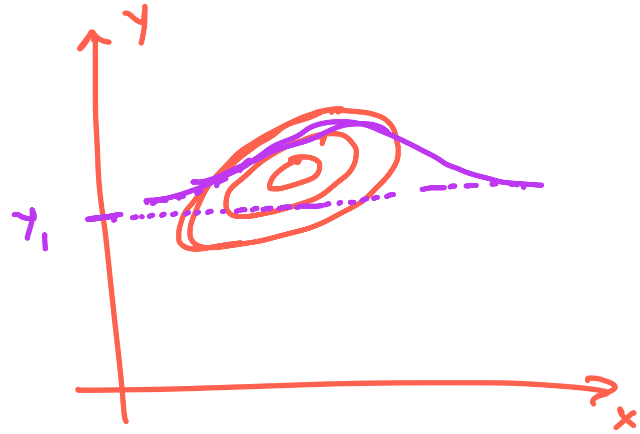
Example : normal distribution

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \begin{matrix} \} \mu_S \\ \\ \} \mu_T \end{matrix} \quad , \quad \Sigma = \begin{pmatrix} \Sigma_{SS} & \Sigma_{ST} \\ \Sigma_{TS} & \Sigma_{TT} \end{pmatrix}$$

If $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim N(\mu, \Sigma)$, then the conditional distributions of $X_S = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$ conditioned on $X_T = \begin{pmatrix} x_{k+1} \\ \vdots \\ x_n \end{pmatrix}$ is given by

$$P_{X_S|X_T} \sim N\left(\mu_T + \Sigma_{ST} \Sigma_{TT}^{-1} (X_S - \mu_T), \right.$$
$$\left. \Sigma_{TT} - \Sigma_{ST}^T \Sigma_{SS}^{-1} \Sigma_{ST} \right)$$

marginal
(collapsing)



conditional
(slicing)

Conditional Expectation

Def (discrete case): $X, Y : (\Omega, \mathcal{A}, P) \to \mathbb{R}$. Assume X takes finitely (countably) many values $x_1, x_2 \dots x_n \in \mathbb{R}$, Y takes finety (conntably) many values $y_1, \dots, y_m \in \mathbb{R}$. Always with positive probability.

$$E(Y \mid X = x_i) := \sum_{j=1}^{m} y_j \underbrace{P(Y = y_j \mid X = x_i)}_{\text{well-defined}}$$

Example: two dice, X = value of die 1, Y = value of die 2, independent dice.

$$E(\text{sum} \mid X = 1) = \sum_{i=1}^{12} i \cdot P(\text{sum} = i \mid X = 1)$$

$$= \sum_{k=1}^{6} (1+k) \cdot P(Y = k \mid X = 1)$$

$$= \sum_{k=1}^{6} (1+k) \cdot P(Y = k) = \sum_{k=1}^{6} (1+k) \cdot \frac{1}{6} = 4.5$$

So far we defined $E(Y|X=x_i)$, but often we want to consider the "function" $E(Y|X)(\omega)$.

This is a RV: $E(Y|X):(\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

This leads to the following:

Def (discrete case): X, Y as before. Then the conditional expectation is defined as follows:

$$E(Y|X) := f(X) \quad \text{with}$$

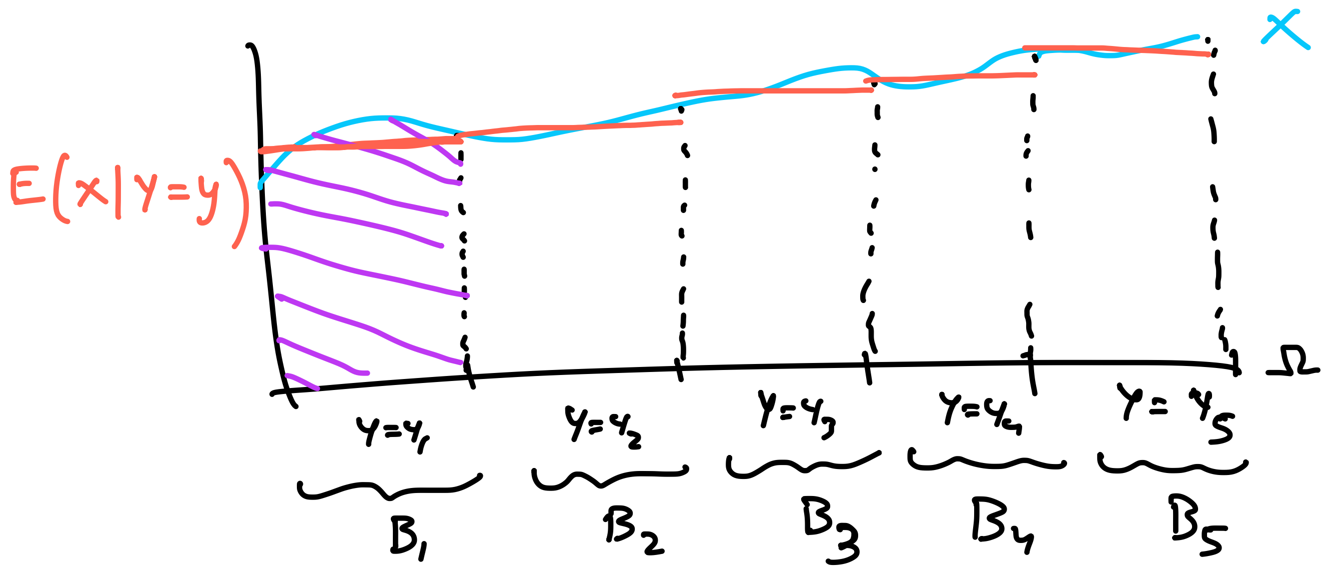
$$f(x) = \begin{cases} E(Y|X=x) & \text{if } P(X=x)>0 \\ \text{arbitrary, say } 0 & \text{otherwise} \end{cases}$$

⚠ $E(Y|X)$ is only defined a.s.

Now we want to consider the more general case.

Sketch: X continuous RV
Y discrete RV $\sim y_1, y_2 \dots y_5$

We want to look at $E(X|Y)$.



Want to "define" $E(X|Y) := \sum_{i=1}^{5} E(X|Y=y_i) \cdot \mathbb{1}_{B_i}(\omega)$

But need to make sure that it is measurable w.r.t. $\sigma(Y)$ (the "bins")

Def (conditional expectation on L_1)

Consider RV $X : (\Omega, \mathcal{A}_0, P) \to \mathbb{R}$, $X \in L_1(\Omega, \mathcal{A}_0, P)$.

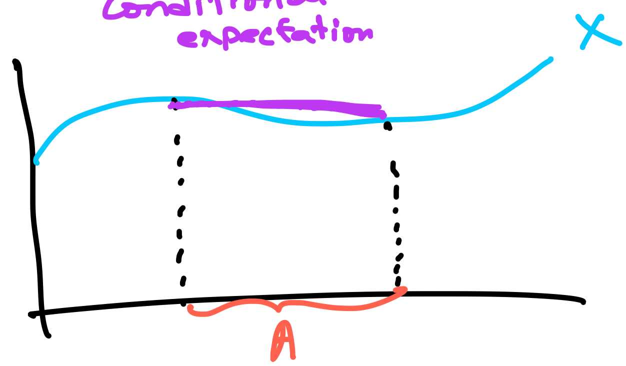
Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{A}_0$. (intuition: $\mathcal{A}$ will be the σ-algebra generated by the variable Y we want to condition on).

We now define the condition expectation of X given $\mathcal{A}$ $E(X|\mathcal{A})$ as any random variable Z that satisfies

(1) Z is measurable w.r.t. $\mathcal{A}$.

(2) For all $A \in \mathcal{A}$ we have

$$\underbrace{\int_A X\,dP} = \underbrace{\int_A Z\,dP}_{\text{conditional expectation}}$$



- Existence of $E(X|\mathcal{A})$ is not clear a priori it needs to be proved.

- $E(X|Y) := E(X|\sigma(Y))$

Examples (extreme cases):

- $X = Y$. Then $E(X|Y) = E(X)$ (a.s.)

- $X \perp\!\!\!\perp Y$. $E(X|Y) = E(X)$ (a.s.)

Case of joint densities

$X, Z : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ have a joint density $f(x, z)$. Let $g : \mathbb{R} \to \mathbb{R}$ bounded, set $Y := g(Z)$. Assume we want to compute $E(Y|X) = E(\underbrace{g(Z)}_{Y}|X)$.

Recall X has density $f_X(x) = \int f(x,z)\,dz$

The conditional density of Z given $X=x$ is

$$f_{X=x}(z) = \frac{f(x,z)}{f_X(x)} \quad (\text{if } f_X(x) \neq 0)$$

Now consider $h(x) := \int \underbrace{g(z)}_{Y} f_{X=x}(z)\,dz$, now define $E(Y|X) = h(X)$.