Transpose

Def: Given a matrix $A = (a_{ij})_{ij} \in F^{m\times n}$, the transpose matrix is given by

$$(A^T)_{kj} = A_{jk}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

If $F = \mathbb{C}$, then the conjugate transpose matrix is defined as,

$$(A^*)_{ij} = \overline{A}_{ji}$$

$$x = a + ib$$

$$\overline{x} = a - ib$$

Useful in the context of the adjoint of an operator.

Change of Basis

Consider the identity mapping $I : V \to V$, $x \mapsto x$. Assume we fix a basis for V (both in source and target space), then the corresponding matrix defined as follows:

$$M(I, \mathcal{B}, \mathcal{B}) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$$

Now consider $A = \{a_1, a_2 \dots a_n\}$ and $\mathcal{B} = \{b_1, b_2 \dots b_n\}$ both bases of V.

How does the matrix of the identity mapping

$$I : (V, A) \to (V, \mathcal{B})$$ look like?

Since $\mathcal{B}$ is a basis, we can express the vectors in A as a linear combination

$$\boxed{a_1 = t_{11} b_1 + t_{21} b_2 + \dots + t_{n1} b_n}$$

$$a_2 = \dots \dots$$

Now we form the corresponding matrix, $T = \begin{pmatrix} \boxed{\begin{matrix} t_{11} \\ \vdots \\ t_{n1} \end{matrix}} & \begin{matrix} \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} t_{1n} \\ \\ t_{nn} \end{matrix} \end{pmatrix}$ $\underline{n \times n}$

This matrix reprents the identity:

- In the basis $\mathcal{A}$, the first basis vector a_1, has the represention $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$a_1 = 1 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n$$

- $T \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} t_{11} \\ t_{21} \\ \vdots \\ t_{n1} \end{pmatrix}$ This vector gives us Ta_1 expressed in the basis $\mathcal{B}$.

- $t_{11} b_1 + t_{21} b_2 + \cdots t_{n1} b_n = a_1$

- $Ta_1 = [a_1]_{\mathcal{B}}$

Proposition: Let $\mathcal{A}, \mathcal{B}$ be two bases of V. Then the matrices $M(I, \mathcal{A}, \mathcal{B})$ and $M(I, \mathcal{B}, \mathcal{A})$ are invertible and each is the inverse of the other.

$$T_{A \to B} = T^{-1}_{B \to A} \qquad T^{-1} \text{ exists.}$$

Proposition: Let $\mathcal{A}, \mathcal{B}$ be two bases of V. Consider the transformation matrix $A = M(I, \mathcal{A}, \mathcal{B})$ and $A^{-1} = M(I, \mathcal{B}, \mathcal{A})$. Let $T : V \to V$ be a linear map, and $X : M(T, \mathcal{A}, \mathcal{A})$. Then $\boxed{Y := A \cdot X \cdot A^{-1}}$ represents T in basis $\mathcal{B}$, that is

$$Y = M(T, \mathcal{B}, \mathcal{B}).$$

$$
\begin{tikzcd}
(V, \mathcal{A}) \arrow[d, "\text{mapping } T"', "\text{matrix } X"] & (V, \mathcal{B}) \arrow[l, dashed, "\text{Identity map}"', "\text{matrix } A^{-1}"] \arrow[d, "\text{map } T"', "\text{matrix } Y"] \\
(V, \mathcal{A}) \arrow[r, dashed, "\text{Id. map}", "\text{matrix } A"'] & (V, \mathcal{B})
\end{tikzcd}
$$

$$
Y = A X A^{-1}
$$

Rank of a Matrix

Def: $A \in F^{m \times n}$. The column rank of A is $\dim(\operatorname{span}(\text{column vectors of } A))$

The row rank is $\dim(\operatorname{span}(\text{row vectors of } A))$

Proposition: For a matrix, the row and column rank are always the same. We now call it the rank of the matrix.

Proposition: $T \in \mathcal{L}(V, W)$. Then

$$\operatorname{rank}(M(T)) = \dim(\operatorname{range}(T))$$

we did not specify any bases

Result holds independent of choice of basis.

Determinant of a Matrix

Def: Consider a linear mapping $d: F^{n\times n} \to F$. Then d is called a <u>determinant</u> if:

(P1) d is multilinear i.e. linear in each column of the matrix:

Let A be a matrix with columns $a_1, a_2 \dots a_n$. Consider column $\begin{pmatrix} | & | & \\ a_1 & a_2 & \dots \\ | & | & \end{pmatrix}$ a_i, assume $a_i = a_i' + a_i''$ for some $a_i', a_i'' \in F^{n\times n}$. Then it holds that

- $\det\left((a_1, \dots, a_i, \dots a_n)\right) = \det\left((a_1, \dots a_i', \dots a_n)\right) + \det\left((a_1 \dots a_i'', \dots a_n)\right)$
- $\det\left((a_1 \dots, \lambda a_i, \dots a_n)\right) = \lambda \det\left((a_1, \dots a_i, \dots a_n)\right)$

(P2) d is alternating: if A has two identical columns, then $\det(A) = 0$

(P3) d is normed: $\det\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1$

Theorem: The mapping d exists and is unique.

Based on (P1), (P2), (P3) we can now prove many important properties of the determinant:

- The determinant of a linear mapping does not depend on the basis.
- $\det(c \cdot A) = c^n \det(A)$; $A \in F^{n \times n}$
- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(A^T) = \det(A)$

- $\det(A^{-1}) = 1/\det(A)$ (if A is invertible)

- A invertible $\iff \det(A) \neq 0$

- $\det(A+B) \neq \det(A) + \det(B)$

- If A is a upper triangular,

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

then $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$

<u>Leibniz Formula</u>: Denote by S_n the set of all permutations of $\{1, 2, \dots n\}$.

then $$\det(A) = \underbrace{\sum_{\sigma \in S_n}}_{\text{all permutations}} \underbrace{\text{sign}(\sigma)}_{\text{sign of a permutation}} a_{1\sigma(1)} \cdot \dots \cdot a_{n\sigma(n)}$$

$\sigma(1)$, $\sigma(n)$: positions in the matrix

Special cases:

$n=1$ $\quad \det(a) = a$

$n=2$ $\quad \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$n=3$ $\quad \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

In general, there exists the formula of Laplace that expresses the determinant of a $n \times n$ matrix as a weighted linear combination of determinants of many $(n-1)\times(n-1)$ sub matrices.

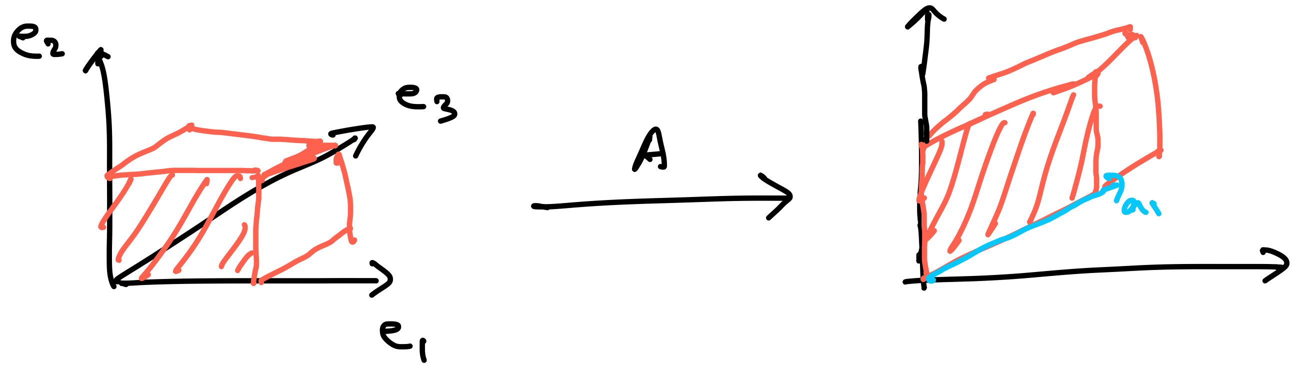
$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} b_{ij} \det(B_{ij})$$

recursive formula

Geometric Intuition

Consider a $n \times n$ matrix A with columns $(a_1, a_2 \dots a_n) = A$. Consider the unit cube

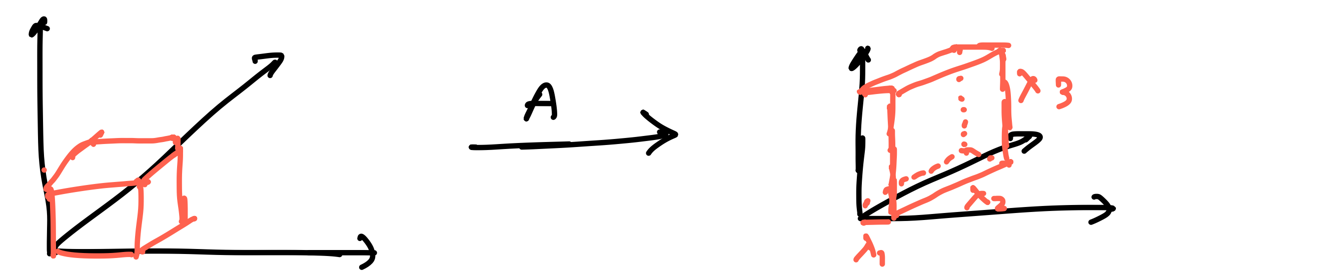
$$U = \{ c_1 e_1 + \dots c_n e_n \mid 0 \leq c_i \leq 1 \}$$



$$U \longmapsto P := \{ c_1 a_1 + c_2 a_2 + \dots c_n a_n \mid 0 \leq c_i \leq 1 \}$$

parallelotope

Then $\det(A)$ gives us the (signed) volume of the parallelotope P



$\mathrm{Vol}(U) = 1$ $\qquad$ $\mathrm{vol}(P) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$\det(A)$ = product of eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A

$\mathrm{vol}(U)$ changes by a factor $\lambda_1 \cdot \lambda_2 \cdot \lambda_3$

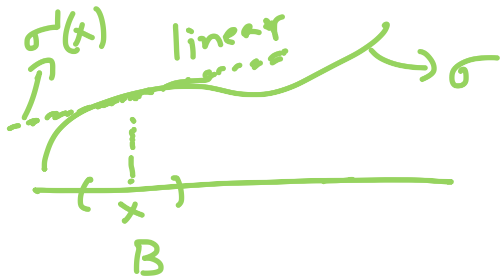
Application to integrals:

Proposition: $\Omega \in \mathbb{R}^n$ open set, $\sigma : \Omega \to \mathbb{R}^n$ differentiable, $f : \sigma(\Omega) \to \mathbb{R}$. Then:

derivative, linear

$$\int_{\sigma(\Omega)} f(y)\,\underbrace{dy}_{\text{volume element}} = \int_{\Omega} f(\sigma(x))\,\underbrace{|\det(\sigma'(x))|}\,dx$$

Intuition: σ differentiable, that is we can locally (on a small ball B around x) approximate σ by a linear function.



$$\sigma'(x) = \begin{pmatrix} \frac{\partial \sigma_1}{\partial x_1} & \cdots\cdots & \frac{\partial \sigma_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial \sigma_n}{\partial x_1} & \cdots\cdots & \frac{\partial \sigma_n}{\partial x_n} \end{pmatrix}$$

$$\begin{aligned} \mathrm{vol}(B) &\approx \mathrm{vol}\big((\sigma'(x)) \cdot B \big) \\ &\approx \left| \det(\sigma'(x)) \right| \cdot \mathrm{vol}(B) \end{aligned}$$

$$f(y) \cdot \underbrace{\mathrm{vol}(B)}_{dy} \approx f(\overbrace{\sigma(x)}^{y}) \cdot \underbrace{\left| \det(\sigma'(x)) \right| \cdot \underbrace{\mathrm{vol}(B)}_{dx}}_{dy}$$

$$\int_{\sigma(\Omega)} f(y)\, dy = \int_{\Omega} f(\sigma(x)) \left| \det \sigma'(x) \right| dx$$