

Transpose

Def: Given a matrix $A = (a_{ij})_{ij} \in F^{m \times n}$,
the transpose matrix is given by

$$(A^T)_{kj} = A_{jk}$$
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$
$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

If $F = \mathbb{C}$, then the conjugate transpose
matrix is defined as,

$$(A^*)_{ij} = \overline{A_{ji}}$$
$$x = a + ib$$
$$\overline{x} = a - ib$$

Useful in the context of the
adjoint of an operator.

Change of Basis

Consider the identity mapping $I: V \rightarrow V$,
 $x \mapsto x$. Assume we fix a basis for V
(both in source and target space), then
the corresponding matrix defined as
follows: $M(I, \mathcal{B}, \mathcal{B}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Now consider $A = \{a_1, a_2, \dots, a_n\}$ and
 $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ both bases of V .
How does the matrix of the identity mapping

$I: (V, A) \rightarrow (V, \mathcal{B})$ look like?

Since \mathcal{B} is a basis, we can express the
vectors in A as a linear combination

$$a_1 = t_{11} b_1 + t_{21} b_2 + \dots + t_{n1} b_n$$

$$a_2 = \dots$$

Now we form the corresponding

matrix, $T = \begin{pmatrix} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \dots & t_{nn} \end{pmatrix}$ $n \times n$

This matrix represents the identity:

• In the basis A , the first basis vector a_1 , has the representation $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$a_1 = 1 \cdot a_1 + 0 \cdot a_2 + \dots + 0 \cdot a_n$$

• $T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} t_{11} \\ t_{21} \\ \vdots \\ t_{n1} \end{pmatrix}$ This vector gives us Ta_1 expressed in the basis B .

• $t_{11} b_1 + t_{21} b_2 + \dots + t_{n1} b_n = a_1$

• $Ta_1 = [a_1]_B$

Proposition: Let A, \mathcal{B} be two bases of V . Then the matrices $M(I, A, \mathcal{B})$ and $M(I, \mathcal{B}, A)$ are invertible and each is the inverse of the other.

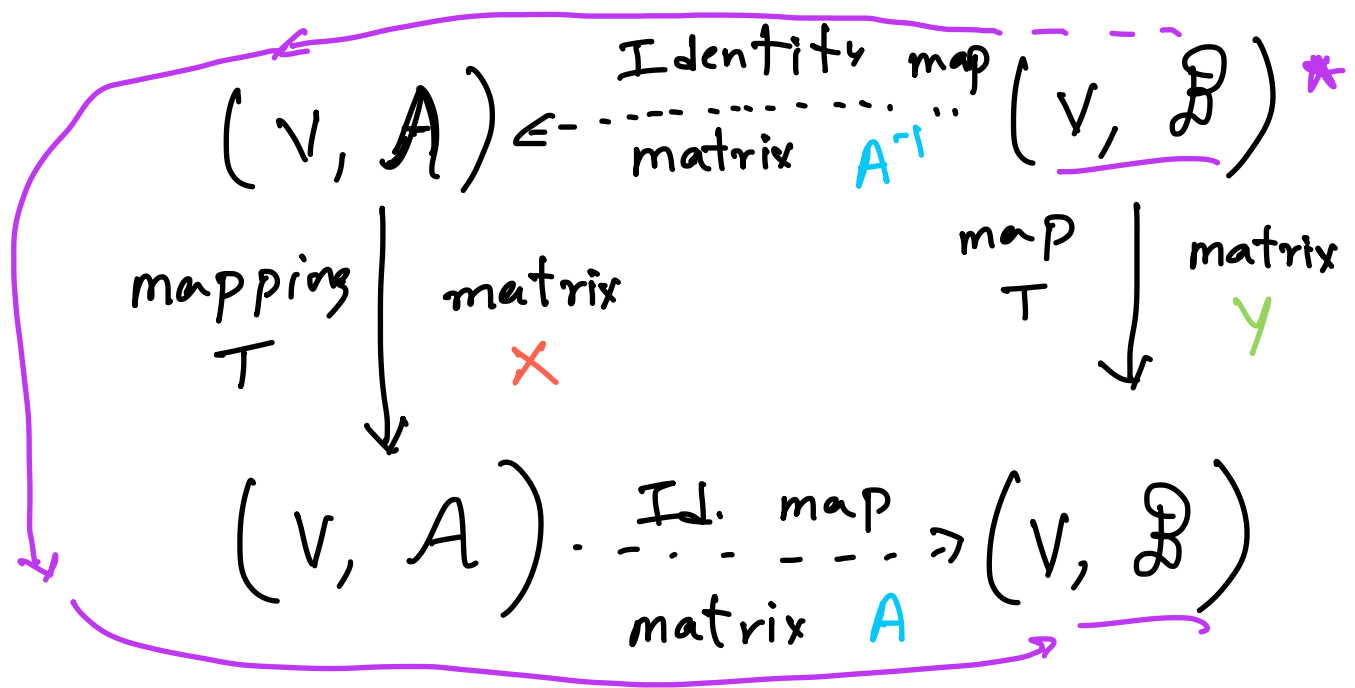
$$T_{A \rightarrow \mathcal{B}} = T_{\mathcal{B} \rightarrow A}^{-1} \quad T^{-1} \text{ exists.}$$

Proposition: Let A, \mathcal{B} be two bases of V . Consider the transformation matrix $A = M(I, A, \mathcal{B})$ and $A^{-1} = M(I, \mathcal{B}, A)$. Let $T: V \rightarrow V$ be a linear map, and

$X := M(T, A, A)$. Then $Y := A \cdot X \cdot A^{-1}$

represents T in basis \mathcal{B} , that is

$$Y = M(T, \mathcal{B}, \mathcal{B}).$$



$$Y = A \times A^{-1}$$

Rank of a Matrix

Def: $A \in F^{m \times n}$. The column rank of A is $\dim(\text{span}(\text{column vectors of } A))$
The row rank is $\dim(\text{span}(\text{row vectors of } A))$

For a matrix, the row and column rank are always the same. We now call it the rank of the matrix.

$T \in \mathcal{L}(V, W)$. Then
 $\text{rank}(M(T)) = \dim(\text{range}(T))$

↓
we did not
specify any
bases

Result holds
independent of choice
of basis.

Determinant of a Matrix

Def: Consider a linear mapping $d: F^{n \times n} \rightarrow F$. Then d is called a determinant if:

(P1) d is multilinear i.e. linear in each column of the matrix:

Let A be a matrix with columns a_1, a_2, \dots, a_n . Consider column a_i , assume $a_i = a_i' + a_i''$ for some $a_i', a_i'' \in F^{n \times n}$. Then it holds that

$$\begin{aligned} & \det(a_1, \dots, a_i, \dots, a_n) = \\ & \det(a_1, \dots, a_i', \dots, a_n) + \det(a_1, \dots, a_i'', \dots, a_n) \\ & \det(a_1, \dots, \lambda a_i, \dots, a_n) = \lambda \det(a_1, \dots, a_i, \dots, a_n) \end{aligned}$$

(P2) d is alternating: if A has two identical columns, then $\det(A) = 0$

(P3) d is normed: $\det \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1$

Theorem: The mapping d exists and is unique.

Based on (P1), (P2), (P3) we can now prove many important properties of the determinant:

- The determinant of a linear mapping does not depend on the basis.
- $\det(c \cdot A) = c^n \det(A)$; $A \in F^{n \times n}$
- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(A^T) = \det(A)$

- $\det(A^{-1}) = 1/\det(A)$ (if A is invertible)

- A invertible $\Leftrightarrow \det(A) \neq 0$

- $\det(A+B) \neq \det(A) + \det(B)$

- If A is an upper triangular,

$$A = \begin{pmatrix} \lambda_1 & \dots & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

then $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

Leibniz Formula: Denote by S_n the set of all permutations of $\{1, 2, \dots, n\}$.

then $\det(A) = \sum_{\sigma \in S_n} \underbrace{\text{sign}(\sigma)}_{\substack{\text{sign of a} \\ \text{permutation}}} a_{1, \underbrace{\sigma(1)}} \cdots a_{n, \underbrace{\sigma(n)}}_{\substack{\text{positions} \\ \text{in the matrix}}}$

all permutations

Special cases:

$$\underline{n=1} \quad \det(a) = a$$

$$\underline{n=2} \quad \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\underline{n=3} \quad \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} \\ - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \\ + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

In general, there exists the formula of Laplace that expresses the determinant of a $n \times n$ matrix as a weighted linear combination of determinants of many $(n-1) \times (n-1)$ sub matrices.

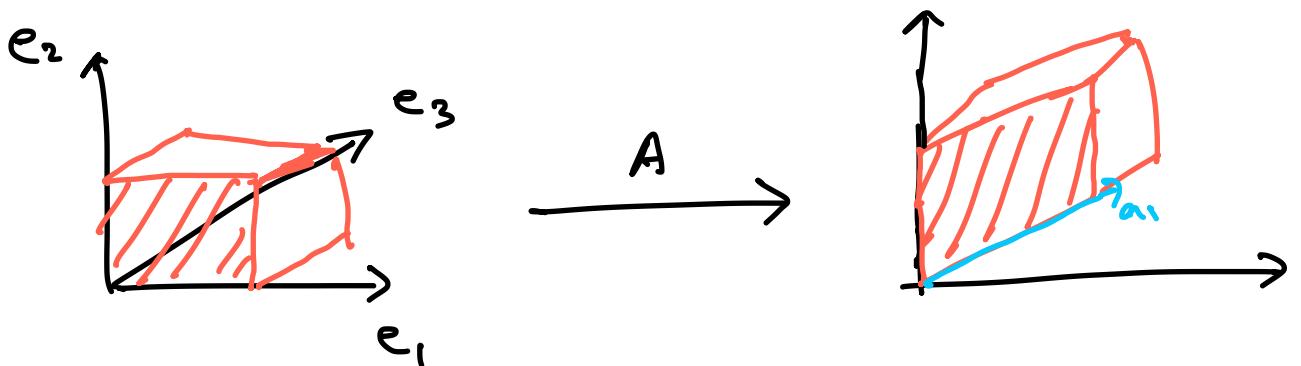
$$\det(A) = \sum_{j=1}^n (-1)^{i+j} b_{ij} \det(B_{ij})$$

recursive formula

Geometric Intuition

Consider a $n \times n$ matrix A with columns
 $(a_1, a_2, \dots, a_n) = A$. Consider the unit cube

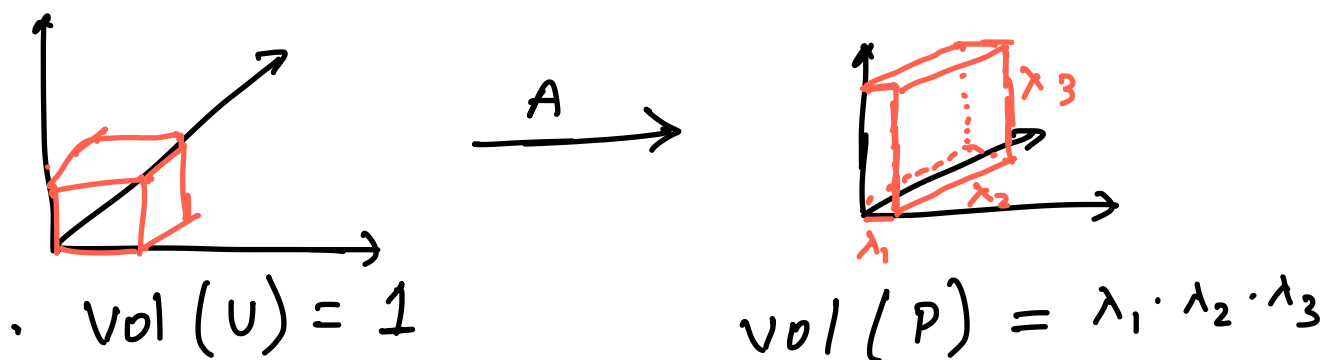
$$U = \{ c_1 e_1 + \dots + c_n e_n \mid 0 \leq c_i \leq 1 \}$$



$$U \mapsto P := \{ c_1 a_1 + c_2 a_2 + \dots + c_n a_n \mid 0 \leq c_i \leq 1 \}$$

parallelepiped

Then $\det(A)$ gives us the (signed) volume
of the parallelepiped P



$\det(A) = \text{product of eigenvalues}$
 $\lambda_1, \lambda_2, \lambda_3 \text{ of } A$

$\text{Vol}(U)$ changes by a factor $\lambda_1 \cdot \lambda_2 \cdot \lambda_3$

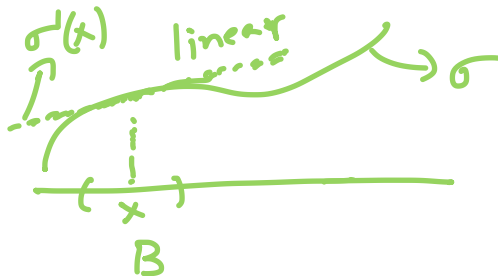
Application to integrals:

$\Omega \in \mathbb{R}^n$ open set, $\sigma: \Omega \rightarrow \mathbb{R}^n$

differentiable, $f: \sigma(\Omega) \rightarrow \mathbb{R}$. Then:

$$\int_{\sigma(\Omega)} \underbrace{f(y)}_{\substack{\downarrow \\ \text{Volume} \\ \text{element}}} dy = \int_{\Omega} f(\sigma(x)) \underbrace{|\det(\sigma'(x))|}_{\substack{\downarrow \\ \text{derivative,} \\ \text{linear}}} dx$$

Intuition: σ differentiable, that is we can locally (on a small ball B around x) approximate σ by a linear function.



$$\sigma'(x) = \begin{pmatrix} \frac{\partial \sigma_1}{\partial x_1} & \dots & \frac{\partial \sigma_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial \sigma_n}{\partial x_1} & \dots & \frac{\partial \sigma_n}{\partial x_n} \end{pmatrix}$$

$$\text{vol}(B) \approx$$