

Eigenvalues

Def: Let $T: V \rightarrow V$. A scalar $\lambda \in F$ is called an eigenvalue if there exists a $v \in V, v \neq 0$, such that $Tv = \lambda v$. A vector $v \neq 0$ with this property is called an eigenvector corresponding to the eigenvalue λ . The set of all eigenvectors of λ is called the eigenspace. $E(\lambda, T) = \text{Ker}(T - \lambda I)$

Remarks:

• Eigenvalue / eigenvector realizes

a "scaling". $v \mapsto \lambda v$

direction of vector v does not change.

• Many linear mappings do not have

eigenvectors. e.g. rotation.

↳ Not true, for algebraically closed fields like \mathbb{C} .

\mathbb{R} is not algebraically closed

$$Tv = \lambda v$$

$$Tv - \lambda v = 0$$

$$(T - \lambda I)v = 0$$

- If λ is an eigenvalue, it has many eigenvectors.

e.g. if v is an eigenvector, then any $a \cdot v$ ($a \in K$) is an eigenvector.

$$T(a \cdot v) = a \cdot T v = a \cdot \lambda v = \lambda (a \cdot v)$$

$$\Rightarrow T(\underline{a \cdot v}) = \lambda (\underline{a \cdot v})$$

- Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Intuition: λ_1, λ_2 two distinct eigenvalues.
i.e. $\lambda_1 \neq \lambda_2$.

Assume v_1, v_2 are eigenvectors that are not linearly independent. i.e. $v_2 = c \cdot v_1$

$$T v_1 = \lambda_1 v_1$$

$$\underline{T v_2} = \lambda_2 v_2 = \lambda_2 (c \cdot v_1)$$

equal

$$\underline{T v_2} = T(c \cdot v_1) = c \cdot T v_1 = c \cdot \lambda_1 v_1$$

not equal

contradiction!!

- Eigenvectors that correspond to the same eigenvalue do not need to be independent.
 e.g. v eigenvector $\Rightarrow c \cdot v$ is also an eigenvector. but v & $c \cdot v$ are not linearly independent.

They can be linearly independent:

e.g. $A = I$, all eigenvalues are 1.

$I \cdot v = 1 \cdot v$. But eigenvectors v can be linearly independent.

- The eigenspace $E(\lambda, T)$ is always a linear subspace of V .

Proposition: For finite-dimensional vector space, the following statements are equivalent:

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|------------------------------------|--|--------------------------------------|
| (i) λ eigenvalue of T | | (iii) $T - \lambda I$ not surjective |
| (ii) $T - \lambda I$ not injective | | (iv) $T - \lambda I$ not bijective |

Proposition: Suppose V is a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T . Then a sum of eigenspaces $E(\lambda_1, T) + E(\lambda_2, T) + \dots + E(\lambda_m, T)$ is a direct sum. In particular

$$\dim(E(\lambda_1, T)) + \dots + \dim(E(\lambda_m, T)) \leq \dim V.$$

Theorem: Every operator $T: V \rightarrow V$ on a finite dimensional vector space with a complex field has at least one eigenvalue.

Proof: Let $n = \dim V$. Choose a vector $v \in V, v \neq 0$. Then the set v, Tv, T^2v, \dots, T^nv has to be linearly dependent. (since it consist of $n+1$ vectors in a n -dim vector space)

Find coefficients a_0, a_1, \dots, a_n such that

$$a_0v + a_1Tv + a_2T^2v + \dots + a_nT^nv = 0$$

Now consider a polynomial on \mathbb{C} with the same coefficients. $P(z) := a_0 + a_1 z + \dots + a_n z^n$

Over \mathbb{C} , we can factorize polynomial as:

$$P(z) = c \cdot (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_m)$$

$m \leq n$

Consider again: $a_0 v + a_1 T v + \dots + a_n T^n v = 0$

$$\Rightarrow (a_0 + a_1 T + \dots + a_n T^n) v = 0$$

$$\Rightarrow c \cdot (T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_m I) v = 0$$

(factorization of polynomial \Rightarrow factorization of operator)

$\Rightarrow v \in \text{Ker}(\textcircled{*})$ (injective $\Leftrightarrow \text{Ker}\{0\}$)

$\Rightarrow \textcircled{*}$ is not an injective function.

\Rightarrow there exists $i \in \{1, \dots, m\}$ such that

$(T - \lambda_i I)$ is not injective

$\Rightarrow \lambda_i$ is an eigenvalue of $T!$ \square

' v ' is not necessarily the eigenvector of T .

Characteristic Polynomial

Motivation: $A\upsilon = \lambda\upsilon$ A $n \times n$ matrix

$$\Rightarrow (A - \lambda I)\upsilon = 0 \quad \upsilon \neq 0$$

$$\Rightarrow \upsilon \in \text{Ker}(A - \lambda I)$$

$$\Rightarrow \text{rank}(A - \lambda I) < n$$

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{image}(T))$$

$$\dim(\text{Ker}(A - \lambda I)) \geq 1$$

$$\Rightarrow \det(A - \lambda I) = 0$$

Def: The characteristic polynomial of a $n \times n$ matrix A is defined as

$$P_A(t) := \det(A - tI)$$

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\det(A - tI) = \det \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} a_{11} - t & a_{12} \\ a_{21} & a_{22} - t \end{pmatrix}$$

$$= (a_{11} - t)(a_{22} - t) - a_{12}a_{21}$$

$$= t^2 - t(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21}$$

Observations:

- $P_A(t)$ is a polynomial with degree n if A is a $n \times n$ matrix.
- Characteristic polynomials do not depend on the choice of basis.

Proof: Consider A , basis transformation matrix U . Want to check if the characteristic polynomial for matrix A and UAU^{-1} are the same.

$$\det(UAU^{-1} - t\underline{I})$$
$$= \det(UAU^{-1} - t\underline{UU^{-1}})$$

$$= \det(UAU^{-1} - tUU^{-1})$$

$$= \det(U(A - tI)U^{-1})$$

$$= \det(U) \cdot \det(A - tI) \cdot \det(U^{-1})$$

$$= \det(A - tI)$$

- The roots of the characteristic polynomial correspond exactly to the eigenvalues of A .
- Over \mathbb{C} , the characteristic polynomial always has n roots, so the matrix has "n eigenvalues" (not necessarily unique).
- A is invertible $\iff 0$ is not an eig. val.
If 0 is an eig. val. $A\underline{v} = 0 \cdot \underline{v} = 0$, $\underline{v} \neq 0$
 $\implies \text{Ker}(A)$ is non-trivial $\iff A$ not invertible.

• Let $A \in \mathcal{L}(V)$, λ is a eig. val. of A .
Then λ^R is an eig. val. of A^R .

↳ geometrically applying A twice will stretch the eig. vec. twice: $\lambda \cdot \lambda = \lambda^2$

• Let A be invertible, λ be eig. val. of A . Then $1/\lambda$ is an eig. val. of A^{-1}

↳ geometrically inverse is unscaling $\rightarrow 1/\lambda$

Def: For an operator A with eigenvalue λ , we define its geometric multiplicity as the dimension of the corresponding eigenspace $E(\lambda, A)$.

Def: The algebraic multiplicity is the multiplicity of the root λ in the characteristic polynomial.

In general, these two notions are not the same

Computing Eigenvalues

- Write down the characteristic polynomial, find the roots \rightarrow eigenvalues.

compute eigenvectors, solve the linear system: $Ax = \lambda x$