

Diagonalization

Def: An operator $T \in \mathcal{L}(V)$ is diagonalizable if there exists a basis of V such that the corresponding matrix is diagonal:

$$M(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Nice property: Diagonal form is the best since we have the eigenvectors as the basis.

Proposition: Let V be a finite-dimensional vector space. $A \in \mathcal{L}(V)$. Then the following statements are equivalent:

(P1) A is diagonalizable

(P2) The characteristic polynomial P_A can be decomposed into linear factors

AND

The algebraic multiplicity of the roots of P_A are equal to the geometric multiplicity

(P3) If $\lambda_1, \dots, \lambda_k$ are the pairwise distinct eigenvalues of A , then

$$V = E(A, \lambda_1) \oplus E(A, \lambda_2) \dots \oplus E(A, \lambda_k)$$

Triangular Matrices

A matrix is called upper triangular, if it has the form

$$\begin{pmatrix} \lambda_1 & \dots & * \\ 0 & \dots & \\ & & \dots \\ & & & \lambda_n \end{pmatrix}$$

Proposition: $T \in \mathcal{L}(V)$, $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ a basis, then following are equivalent:

(P1) $M(T, \mathcal{B})$ is upper triangular

(P2) $Tv_j \in \text{span} \{v_1, v_2, \dots, v_j\}$
 $\forall j = 1, 2, \dots, n$

$$Tv_1 = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \cdot v_1$$

$$Tv_2 = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ \lambda_2 \\ 0 \end{pmatrix} = a_{12} \overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{v_1} + \lambda_2 \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{v_2} \in \text{span}(v_1, v_2)$$

Proposition: V complex, finite-dim VS ,

$T \in \mathcal{L}(V)$. Then $M(T)$ has an upper triangular form for some basis.

→ If we are in the complex field, every matrix can be expressed as an upper triangular matrix.

Proposition: Suppose $T \in \mathcal{L}(V)$, V any finite-dim VS , has an upper triangular form. Then the entries on the diagonal are precisely the eigenvalues of T .

Metric Space

Metric spaces \rightarrow Normed spaces \rightarrow inner product spaces \rightarrow Hilbert spaces

K-NN \rightarrow metric

Def: Let X be a set. A function $d: X \times X \rightarrow \mathbb{R}$ is called a metric if the following conditions hold. $\forall u, v, w \in X$:

(P1) $d(u, v) > 0$ if $u \neq v$
and $d(u, u) = 0$

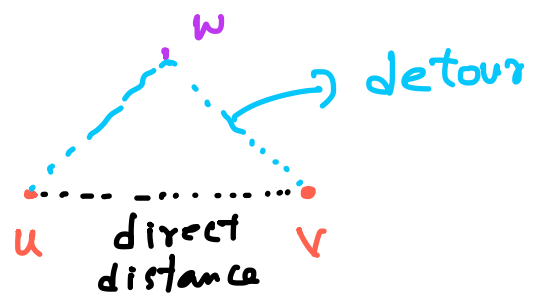
(P2) $d(u, v) = d(v, u)$
(symmetry)

(P3) $d(u, v) \leq d(u, w) + d(w, v)$



Eg: asymmetric measures

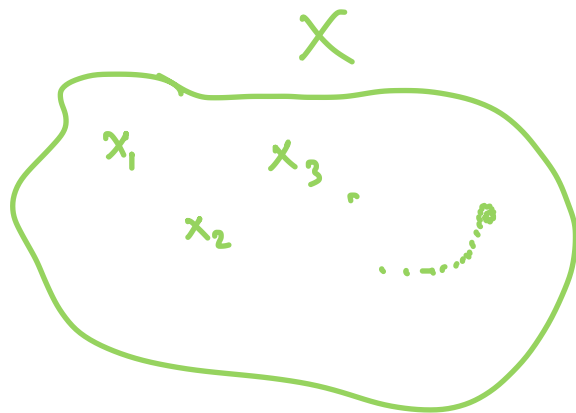
- (i) friendship graph
- (ii) $KL(P||Q) \neq KL(Q||P)$



Sequence: $(x_1, x_2, \dots) \rightarrow (x_n)_{n \in \mathbb{N}}$

Def: A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is called a Cauchy Sequence if $\forall \varepsilon > 0 \exists N \in \mathbb{N}, \forall n, m > N, d(x_n, x_m) < \varepsilon$

Eg:



Def: A sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N, d(x_n, x) < \varepsilon$

Notation: $x_n \rightarrow x, \lim_{n \rightarrow \infty} x_n = x$

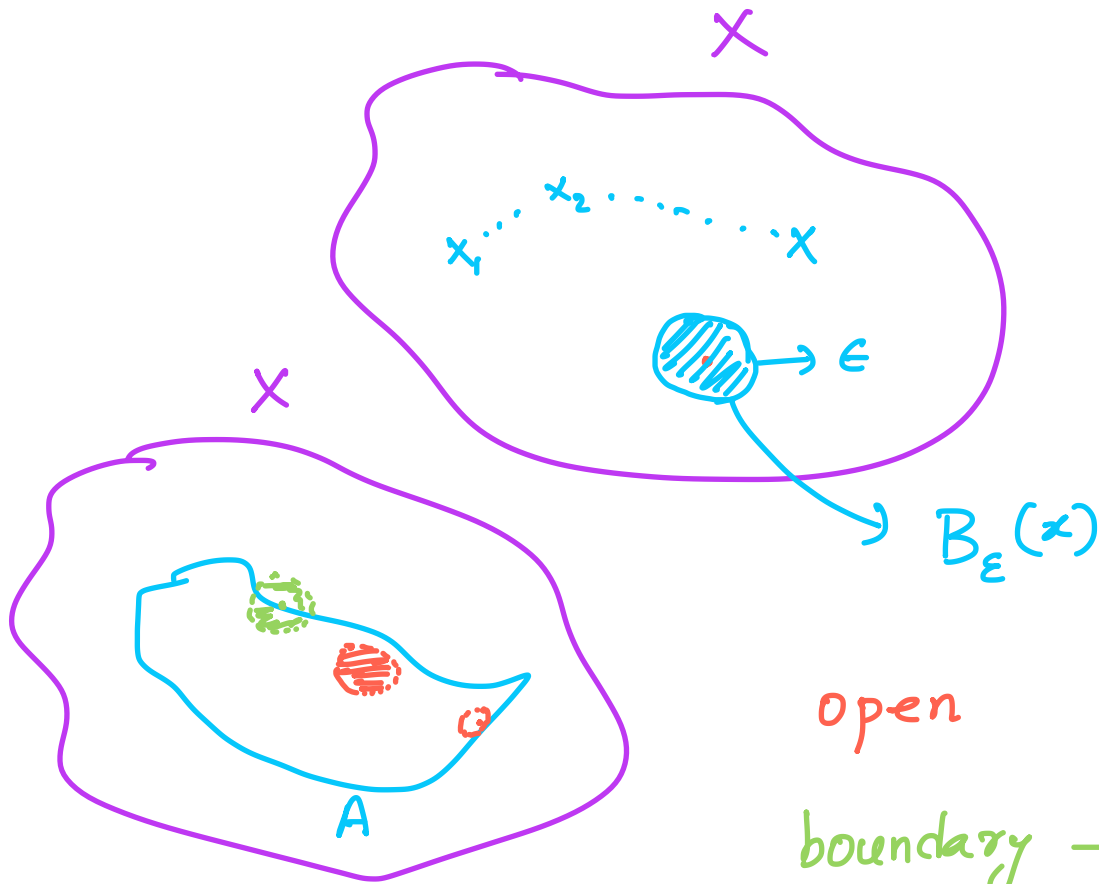
Sequence $(x_n)_{n \in \mathbb{N}} = 1/n$ on $X = (0, 1)$

Here $(x_n)_{n \in \mathbb{N}}$ is a Cauchy seq. $\frac{1}{n} - \frac{1}{m}$

but does not converge.

Sequence $(x_n)_{n \in \mathbb{N}} = 1/n$ on $\tilde{X} = [0, 1]$. Here $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence that converges on \tilde{X} to 0.

Def: A metric space is called complete if every Cauchy sequence converges.



open $A \subseteq X$

boundary $\rightarrow \partial A$

closed $:= A \cup \partial A$

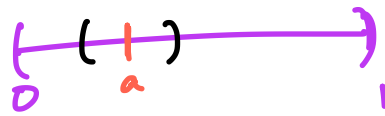
Notation: $B_\epsilon(u) := \{x \in X \mid d(x, u) \leq \epsilon\} \rightarrow \epsilon\text{-ball}$

Def: A set $A \subseteq X$ is called closed if all Cauchy sequences converge and have their limit point of A

Def: A set $A \subset X$ is called open if

$$\forall a \in A \exists \varepsilon > 0: B_\varepsilon(a) \subset A$$

- Set $[0, 1]$ is closed
- Set $(0, 1)$ is open



$$B = (a - \varepsilon, a + \varepsilon)$$

- A set A can be neither open nor closed. e.g. $[0, 1)$

Def: A point $a \in A$ is an interior point of A if $\exists \varepsilon > 0$, s.t. $B_\varepsilon(a) \subset A$

e.g. $A = [0, 1]$, then $x \in (0, 1)$ are interior points.

Def: The (topological) closure of a set A is defined as the set of points that can be approximated by Cauchy sequences in A :

$$w \in \bar{A} \iff \forall \varepsilon > 0 \exists z \in A: d(w, z) < \varepsilon$$

Notation: \bar{A} is the closure of A . $A \cup \partial A$
(always closed!)

Def: The (topological) interior of a set A is defined as the set of interior points of A .

Notation: A°

Def: The (topological) boundary of a set A is defined as the set $\bar{A} \setminus A^\circ$

$$X = [0, 1] \quad \text{sometimes}$$

$$\bar{X} = [0, 1] \quad \partial X = X \setminus X^\circ$$

$$X^\circ = (0, 1) \quad = \{0\}$$

$$\Rightarrow \text{boundary } \partial X = \bar{X} \setminus X^\circ = \{0, 1\}$$

Def: A set A is dense in X if we can approximate every $x \in X$ by a sequence in A .

Formally, $\forall x \in X \quad \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$

Example: $\mathbb{Q} \subset \mathbb{R}$ is dense

Def: A set $A \subset X$ is bounded if there exists $D > 0$ such that $\forall u, v \in A$

$$d(u, v) < D.$$

Norms

Def: Let V be a vector space. A norm on V is a function $\|\cdot\|: V \rightarrow \mathbb{R}$ such that $\forall x, y \in V, \lambda \in F$, the following conditions hold:

$$(P1) \quad \|\lambda x\| = |\lambda| \|x\| \quad (\text{homogeneous})$$

$$(P2) \quad \|x + y\| \leq \|x\| + \|y\| \quad (\text{triangle inequality})$$

$$(P3) \quad x = 0 \Rightarrow \|x\| = 0$$

$$(P4) \quad \|x\| = 0 \Rightarrow x = 0$$

$\|\cdot\|$ is a semi-norm if (P1)-(P3) are satisfied.

Intuition: $\text{norm}(x) =$ "length of x "
 $=$ distance $(x, 0)$

Example: Euclidean norm on \mathbb{R}^d : $\|x\| = \left(\sum_{i=1}^d x_i^2 \right)^{1/2}$

Manhattan distance: $\|x\| = \left(\sum_{i=1}^d |x_i| \right)$

p-Norm

Consider $V = \mathbb{R}^d$. Define $\|\cdot\|_p : \mathbb{R}^d \rightarrow \mathbb{R}$

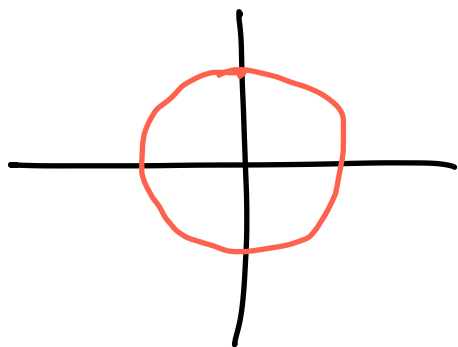
$$\|x\|_p := \left(\sum_{i=1}^d |x_i|^p \right)^{1/p} \text{ for } 0 < p < \infty$$

• $\|\cdot\|_p$ is a norm if $p \geq 1$

• Unit balls: the unit ball of a norm is the set of points such that norm ≤ 1 :

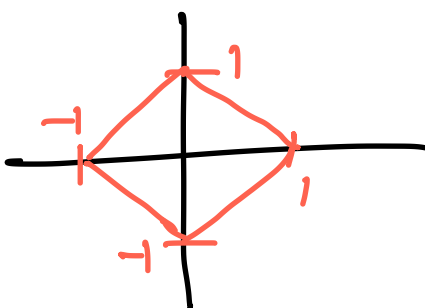
$$B_p := \{ x \in \mathbb{R}^d \mid \|x\|_p \leq 1 \}$$

Examples: \mathbb{R}^2



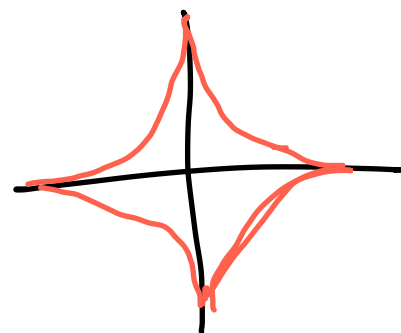
$p=2$

(convex)



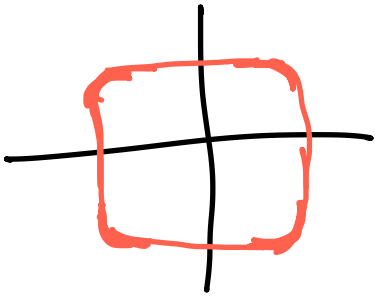
$p=1$

(convex)

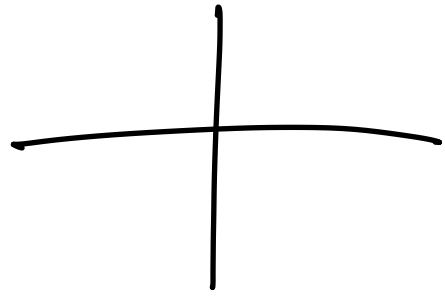


$p=0.5$

(not convex)



$$p = 5$$



$$p = 0.1$$

$$p = \infty$$

$$p = 0$$