Diagonalization

Def: An operator $T \in \mathcal{L}(V)$ is diagonalizable if there exists a basis of V such that the corresponding matrix is diagonal:

$$M(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Nice property: Diagonal form is the best since we have the eigenvectors as the basis.

Proposition: Let V be a finite-dimensional vector space. $A \in \mathcal{L}(V)$. Then the following statements are equivalent:

(P1) A is diagonalizable

(P2) The characteristic polynomial P_A can be decomposed into linear factors

AND

The algebraic multiplicity of the roots of P_A are equal to the geometric multiplicity

(P3) If $\lambda_1, \dots \lambda_k$ are the pairwise distinct eigenvalues of A, then

$$V = E(A, \lambda_1) \oplus E(A, \lambda_2) \dots \oplus E(A, \lambda_k)$$

Triangular Matrices

A matrix is called upper triangular, if it has the form

$$\begin{pmatrix} \lambda_1 & \cdots & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Proposition: $T \in \mathcal{L}(V)$, $\mathcal{B} = \{v_1, v_2 \dots v_n\}$ a basis, then following are equivalent:

(P1) $M(T, \mathcal{B})$ is upper triangular

(P2) $Tv_j \in \text{span}\{v_1, v_2 \dots v_j\}$ $\forall\, j = 1, 2, \dots n$

$$Tv_1 = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \cdot v_1$$

$$Tv_2 = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ \lambda_2 \\ 0 \end{pmatrix} = a_{12} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + \lambda_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_2}$$

$\in \text{span}(v_1, v_2)$

Proposition: V complex, finite-dim VS, $T \in \mathcal{L}(V)$. Then $M(T)$ has an upper triangular form for some basis.

→ If we are in the complex field, every matrix can be expressed as an upper triangular matrix.

Proposition: Suppose $T \in \mathcal{L}(V)$, V any finite-dim VS, has an upper triangular form. Then the entries on the diagonal are precisely the eigenvalues of T.

Metric Space

Metric spaces → Normed spaces → inner product spaces → Hilbert spaces

K-NN → metric

Def: Let X be a set. A function $d: X \times X \to \mathbb{R}$ is called a metric if the following conditions hold. $\forall u, v, w \in X$:

(P1) $d(u, v) > 0$ if $u \neq v$

and $d(u, u) = 0$

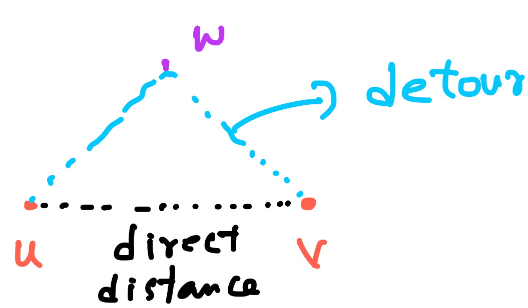
(P2) $d(u, v) = d(v, u)$ (symmetry)

Eg: assymetric measures

(i) friendship graph

(ii) $KL(p \| q) \neq KL(q \| p)$

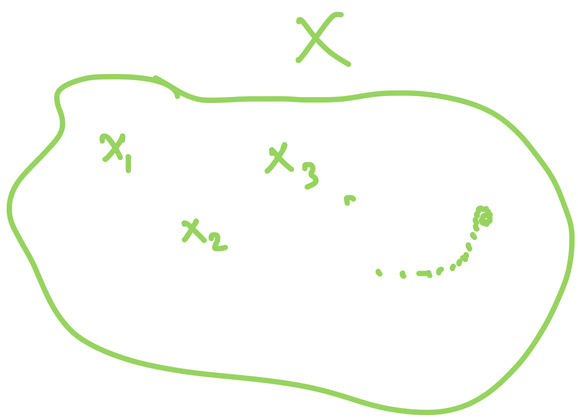
(P3) $d(u, v) \leq d(u, w) + d(w, v)$



<u>Sequence</u>: $(x_1, x_2 \dots) \rightarrow (x_n)_{n\in\mathbb{N}}$

<u>Def</u>: A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is called a <u>Cauchy Sequence</u> if $\forall \varepsilon > 0\ \exists N \in \mathbb{N},\ \forall n, m > N,\ d(x_n, x_m) < \varepsilon$

<u>Eg</u>:



<u>Def</u>: A sequence $(x_n)_{n\in\mathbb{N}}$ <u>converges</u> to $x \in X$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N,\ d(x_n, x) < \varepsilon$

<u>Notation</u>: $x_n \rightarrow x$, $\lim_{n\to\infty} x_n = x$

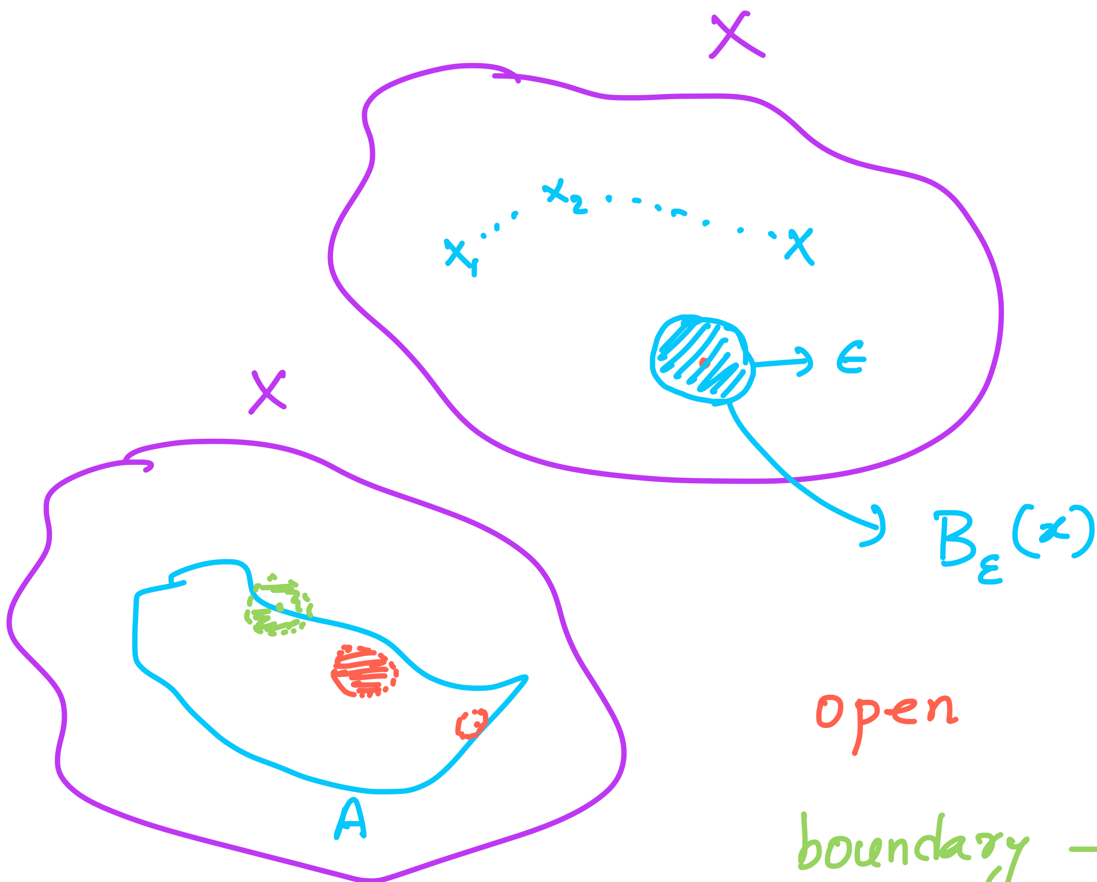
Sequence $(x_n)_{n\in\mathbb{N}} = 1/n$ on $X = (0,1)$

Here $(x_n)_{n\in\mathbb{N}}$ is a Cauchy seq. $\frac{1}{n} - \frac{1}{m}$

but does not converge.

Sequence $(x_n)_{n\in\mathbb{N}} = 1/n$ on $\tilde{X} = [0,1]$. Here $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges on $\tilde{X}$ to 0.

Def: A metric space is called <u>complete</u> if every Cauchy sequence converges.



open $A \subseteq X$

boundary $\to \partial A$

closed $:= A \cup \partial A$

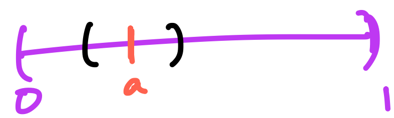
Notation: $B_\epsilon(u) := \{ x \in X \mid d(x,u) \leq \varepsilon \} \to \varepsilon$- ball

Def: A set $A \subseteq X$ is called <u>closed</u> if all Cauchy sequences converge and have their limit point of A

Def: A set $A \subset X$ is called <u>open</u> if

$$\forall a \in A \ \exists \varepsilon > 0 : \quad B_\varepsilon(a) \subset A$$

- Set $[0,1]$ is closed
- Set $(0,1)$ is open



$$B = (a-\epsilon, a+\epsilon)$$

- A set A can be neither open nor closed. e.g. $[0,1)$

Def: A point $a \in A$ is an <u>interior point</u> of A if $\exists \varepsilon > 0$, s.t. $B_\varepsilon(a) \subset A$

e.g. $A = [0,1]$, then $x \in (0,1)$ are interior points.

Def: The (topological) closure of a set A is defined as the set of points that can be approximated by Cauchy sequences in A:

$$w \in \bar{A} \iff \forall \varepsilon > 0 \ \exists z \in A : d(w,z) < \varepsilon$$

<u>Notation</u>: $\bar{A}$ is the closure of A. $A \cup \partial A$ (always closed!)

Def: The (topological) interior of a set A is defined as the set of interior points of A.

Notation: A°

Def: The (topological) boundary of a set A is defined as the set $\bar{A} \setminus A^\circ$

$$X = [0, 1)$$

$$\bar{X} = [0, 1]$$

$$X^\circ = (0, 1)$$

sometimes $\partial X = X \setminus X^\circ$
$= \{0\}$

$\Rightarrow$ boundary $\partial X = \bar{X} \setminus X^\circ = \{0, 1\}$

Def: A set A is dense in X if we can approximate every $x \in X$ by a sequence in A.

Formally, $\forall x \in X \ \forall \epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$

Example: $\mathbb{Q} \subset \mathbb{R}$ is dense

Def: A set $A \subset X$ is bounded if there exists $D > 0$ such that $\forall u, v \in A$

$$d(u, v) < D.$$

Norms

Def: Let V be a vector space. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that $\forall x, y \in V$, $\lambda \in F$, the following conditions hold:

(P1) $\|\lambda x\| = |\lambda| \, \|x\|$ (homogeneous)

(P2) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

(P3) $x = 0 \Rightarrow \|x\| = 0$

(P4) $\|x\| = 0 \Rightarrow x = 0$

$\|\cdot\|$ is a semi-norm if (P1)-(P3) are satisfied.

Intuition: norm(x) = "length of x"

= distance$(x, 0)$

Example: Euclidean norm on $\mathbb{R}^d$: $\|x\| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}$

Manhattan distance: $\|x\| = \left(\sum_{i=1}^{d} |x_i| \right)$

p-Norm

Consider $V = \mathbb{R}^d$. Define $\|\cdot\|_p : \mathbb{R}^d \to \mathbb{R}$

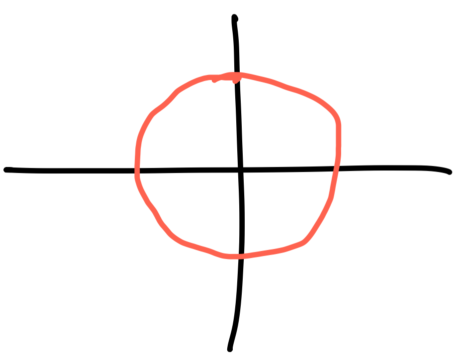
$$\|x\|_p := \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \quad \text{for} \quad 0 < p < \infty$$

- $\|\cdot\|_p$ is a norm if $p \geq 1$

- <u>Unit balls:</u> the unit ball of a norm is the set of points such that norm ≤ 1:

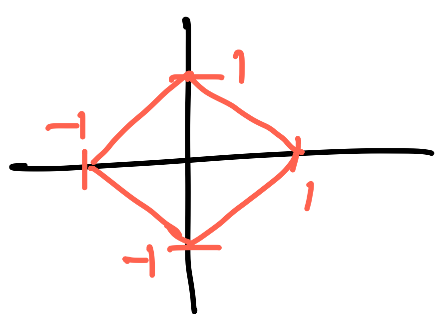
$$B_p := \{ x \in \mathbb{R}^d \mid \|x\|_p \leq 1 \}$$

<u>Examples</u>: $\mathbb{R}^2$



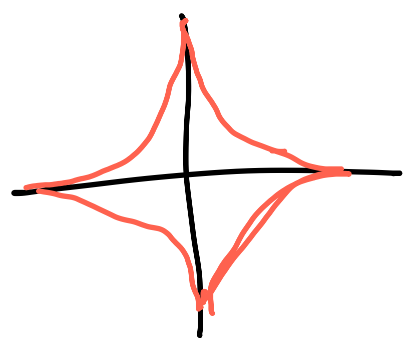
$p=2$

(convex)



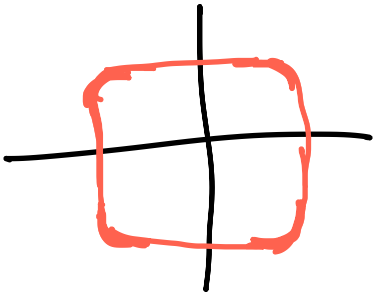
$p=1$

(convex)

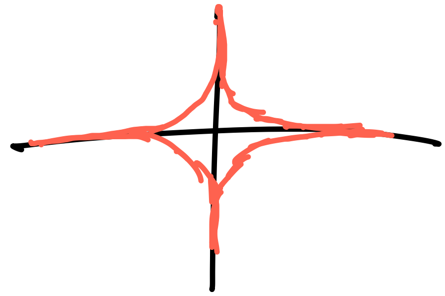


$p=0.5$

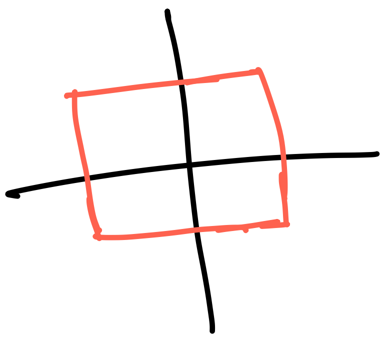
(not convex)



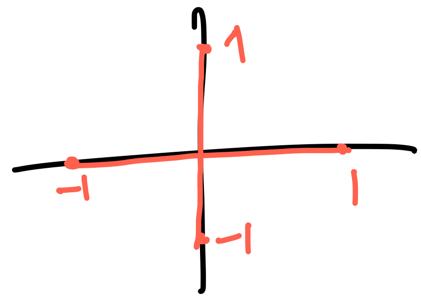
$p = 5$



$p = 0.1$



$p = \infty$



$p = 0$

Def: $\|x\|_\infty := \max |x_i|$ (is a norm)

$\|x\|_0 :=$ number of non-zero coordinates

$$= \sum_{i=1}^{d} \mathbb{1}_{\{x_i \neq 0\}}$$

⚠ $\|x\|_0$ is not a norm

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\|x\|_0 = 1$; $\lambda x = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $\|\lambda x\|_0 = 1$

$\lambda = 5$ $\neq 5 \cdot 1$