**Theorem:** All norms on $\mathbb{R}^n$ are (topologically) equivalent: If $\|\cdot\|_a$ and $\|\cdot\|_b : \mathbb{R}^n \to \mathbb{R}$ are two norms on $\mathbb{R}^n$, then there exists constants $\alpha, \beta > 0$ such that:

$$\forall x \in \mathbb{R}^n : \alpha \|x\|_a \leq \|x\|_b \leq \beta \|x\|_a.$$ 

**Proof:** Without loss of generality (WLOG), we prove that if $\|\cdot\|_1$ is any norm on $\mathbb{R}^n$, then it is equivalent to $\|\cdot\|_\infty$ on $\mathbb{R}^n$.

**First Inequality:** $\exists C_1 > 0 : \forall x : \|x\|_1 \leq C_1 \|x\|_\infty$

Let $x = \sum x_i e_i$ be the representation of $x$ in the standard basis of $\mathbb{R}^n$.

$$\|x\|_1 = \| \sum_{i=1}^n x_i e_i \| \leq \sum_{i=1}^n |x_i| \|e_i\| = \sum_{i=1}^n |x_i| e_i$$

$$\leq \sum_{i} \|x_i\| \|e_i\| = \|x\|_\infty \sum_{i=1}^n \|e_i\| \Rightarrow \|x\|_1 \leq C_1 \|x\|_\infty$$

$= : C_1$
Second Inequality: \( \exists C_2 > 0 \ \forall x \ |x| \leq C_2 \cdot \|x\|_\infty \)

Let \( S := \{ x \in \mathbb{R}^n \mid \|x\|_\infty = 1 \} \) be the unit sphere w.r.t. \( \|\cdot\|_\infty \)

Consider \( f : S \rightarrow \mathbb{R}, \ x \mapsto \|x\| \)

The mapping \( f \) is continuous w.r.t. \( \|\cdot\|_\infty \): this follows from the fact that:

\[
|f(x) - f(y)| = |\|x\| - \|y\|| \leq \|x - y\| \leq C_1 \|x - y\|_\infty
\]

Lipschitz continuity

The set \( S \) is closed and bounded, so \( S \) is compact (from analysis). Any continuous mapping on a compact set takes its min and max.

Define \( \overline{C}_2 := \min \{ f(x) \mid x \in S \} \)

\[
x \in S: \ |\|x\| - \|x\|_\infty| = |\|x\| - \|x\|_\infty| = \frac{|\|x\|_\infty - \|x\|_\infty|}{\|x\|_\infty} = \frac{\|x\|}{\|x\|_\infty}
\]

Since \( x \in S \) \( , \|x\|_\infty = 1 \)

\[
\Rightarrow \overline{C}_2 \leq \frac{\|x\|}{\|x\|_\infty} \Rightarrow \|x\|_\infty \leq \frac{1}{\overline{C}_2} \|x\| \\Rightarrow \overline{C}_2 \leq \frac{\|x\|}{\|x\|_\infty} \leq \frac{1}{\overline{C}_2} \|x\|_\infty \leq \frac{1}{\overline{C}_2} \|x\| \leq \frac{1}{\overline{C}_2} \|x\|_\infty \leq \frac{1}{\overline{C}_2}
\]

\( c_2 := \frac{1}{\overline{C}_2} \)
Convex Sets are Unit balls of norms

Def: Consider a real vector space, $V$. $S$ is called convex if $\forall b \geq 0, b \leq 1$ and $\forall x, y \in S$, $b \cdot x + (1-b) \cdot y \in S$

Intuition:
**Def:** A set $C \subseteq \mathbb{V}$ is called symmetric if $x \in C \Rightarrow -x \in C$

![Diagram](image)

**Theorem:** (1) Let $C \subseteq \mathbb{R}^d$ closed, convex, symmetric and has non-empty interior. Define $p(x) := \inf \left\{ t > 0 \mid x \in t \cdot C \right\}$. Then $p$ is a semi-norm. If $C$ is bounded, then $p$ is a norm, and its unit ball coincides with $C$. (i.e. $C = \{ x \in \mathbb{R}^d \mid p(x) \leq 1 \}$

(2) For any norm $\| \cdot \|$ on $\mathbb{R}^d$, the set $C := \{ x \in \mathbb{R}^d \mid \|x\| \leq 1 \}$ is bounded, symmetric, closed, convex and has non-empty interior.
Proof: \( p(x) \) is well defined

Want to prove: given \( x \in \mathbb{R}^d \), the set
\( \{ t > 0 \mid x \in t \cdot c \} \neq \emptyset \).

We are going to prove: \( \exists \epsilon > 0 \) such that
\( B_\epsilon(0) = \{ y \in \mathbb{R}^d \mid \| y \|_2 < \epsilon \} \subseteq C \)

Intuition:

- By assumption, \( C \) has at least one interior point.
  \( v \in C^\circ \Rightarrow \exists \epsilon \) such that
  \( B_\epsilon(v) \subseteq C \Rightarrow v + B_\epsilon(0) = \{ v + y \mid y \in B_\epsilon(0) \} \subseteq C \).
By symmetry, \( u + e \in C \Rightarrow -(u + e) \in C \)

By convexity, \( \frac{1}{2}(u + e) + \frac{1}{2}(-u - e) = e \in C \)

So \( B_e(0) \subseteq L \), so the set \( \{ t > 0 \mid x \in t \cdot C \} \)

is non-empty.

The infimum of \( \inf \{ t > 0 \mid x \in t \cdot C \} \) exists because \( \{ t > 0 \mid x \in t \cdot C \} \subseteq \mathbb{R} \) has 0 as its lower bound.

\[ (P1) \quad P(0) = 0 \]

- have seen: \( D \subseteq C \)
- \( \forall t > 0: \quad 0 \in o \cdot C \)
- \( \inf \{ t \mid o \in t \cdot C \} = 0 \)

\( \Rightarrow \quad P(0) = 0 \)
\[(P_2) \quad P(\alpha x) = |\alpha| \cdot p(x)\]

. \(\forall \alpha > 0 \), we have

\[
p(\alpha x) = \inf \{ t > 0 \mid \alpha x \in S \cdot c \} = \inf \{ \frac{s > 0}{\alpha} \mid x \in S \cdot c \}
\]

\[
= \alpha \cdot \inf \{ s > 0 \mid x \in S \cdot c \}
\]

\[
= \alpha \cdot p(x)
\]

\[
=) \quad p(\alpha x) = \alpha \cdot p(x)
\]

. By symmetry we also get

\[
p(-x) = p(x)
\]

. Combining the two statements to say \( p(\alpha \cdot x) = |\alpha| \cdot p(x) \)

( homogeneity )
(P3) **Triangle - Inequality.**

Consider $x, y \in \mathbb{R}^d$, $s, t > 0$ such that:

\[
\frac{x}{s} \in C, \quad \frac{y}{t} \in C.
\]

Observe:

\[
\frac{s}{s+t} + \frac{t}{s+t} = 1,
\]

Then, by convexity,

\[
\frac{s}{s+t} \cdot \frac{x}{s} + \frac{t}{s+t} \cdot \frac{y}{t} \in C \implies \frac{x+y}{s+t} \in C
\]

Thus, two scalars that sum to 1

\[
\implies \frac{x+y}{s+t} \in S
\]

\[
\implies P(x+y) = \inf \{ u > 0 \mid x+y \in u \cdot C \} \leq u_0
\]

\[
\leq s + t
\]

\[
= P(x) + P(y)
\]

$s$ was chosen such that $x \in s \cdot C$

$t$ was chosen such that $y \in t \cdot C$
Consider a sequence $\{s_i\}_{i \in \mathbb{N}}$ such that $x \in s_i \cdot C$ and $s_i \rightarrow p(x)$.

Similarly $\{t_i\}_{i \in \mathbb{N}}$ such that $y \in t_i \cdot C$ and $t_i \rightarrow p(y)$.

\[ \forall i : \quad p(x + y) \leq s_i + t_i \]

\[ \Rightarrow \quad p(x + y) \leq p(x) + p(y) \]

\[
(PH) \quad p(x) = 0 \Rightarrow x = 0
\]

\[ p(x) = 0 \iff \inf \{ t > 0 \mid x \in t \cdot C \} = 0 \]

\[ \Rightarrow \text{There exists a sequence } \{t_k\}_{k \in \mathbb{N}} \text{ s.t. } t_k \rightarrow 0 \text{ and } x \in t_k \cdot C \forall k. \]

Now assume that $x \neq 0$. Then the sequence $\left( \frac{x}{t_k} \right)_{k \in \mathbb{N}}$ is unbounded.

\[ \Rightarrow \text{contradiction since } C \text{ is bounded.} \]
**Normed Function Spaces**

**Space of continuous functions**

Let $T$ be a metric space,

$$C^b(T) := \{ f : T \to \mathbb{R} \mid f \text{ is continuous and bounded} \}$$

As norm on $C^b(T)$ we choose:

$$\| f \|_\infty := \sup_{t \in T} |f(t)|$$

The norm exists since we are in the space of bounded functions, bounded from above.

Then the space $C^b(T)$ with norm $\| \cdot \|_\infty$ is a Banach space.

If $(X, d_{\| \cdot \|})$ is a complete metric space, then the normed space $(X, \| \cdot \|)$ is called a Banach space.
Proof outline:
(i) need to check vector space axioms
(ii) norm axioms
(iii) completeness: follows from the fact that \( \| \cdot \|_\infty \) induces uniform convergence.

Space of differentiable functions:
Let \([a, b] \subseteq \mathbb{R}, \mathcal{C}^1([a, b]) = \{f: [a, b] \to \mathbb{R} | f \text{ is continuously differentiable}\} \)

Which norm?
Consider \( \| \cdot \|_\infty \). With this norm, \( \mathcal{C}^1 \) is not complete.

Is there a better norm. The answer is yes. Many norms exist, let us consider a few examples.
Consider \[ \| f \| := \sup_{t \in [a, b]} \max \left\{ |f(t)|, |f'(t)| \right\} \]

Consider \[ \| f \| := \| f \|_{\infty} + \| f' \|_{\infty} \]

\( C'([a, b]) \) with any of these two norms is a Banach space.