Equivalence of Norms

Theorem: All norms on $\mathbb{R}^n$ are (topologically) equivalent: If $\|\cdot\|_a$ and $\|\cdot\|_b : \mathbb{R}^n \to \mathbb{R}$ are two norms on $\mathbb{R}^n$, then there exists constants $\alpha, \beta > 0$ such that:

$$\forall x \in \mathbb{R}^n : \quad \alpha \|x\|_a \leq \|x\|_b \leq \beta \|x\|_a .$$

Proof: Without loss of generality (W.l.o.g) we prove that if $\|\cdot\|$ is any norm on $\mathbb{R}^n$, then it is equivalent to $\|\ \|_\infty$ on $\mathbb{R}^n$.

$\boxed{\text{First Inequality: } \exists c_1 > 0 : \ \forall x \ \ \|x\| \leq c_1 \|x\|_\infty}$

Let $x = \sum x_i e_i$ the representation of x in the standard basis of $\mathbb{R}^n$.

$$\|x\| = \left\| \sum_{i=1}^n x_i e_i \right\| \leq \sum_{i=1}^n \|x_i e_i\| = \sum_i \underbrace{|x_i|}_{\leq \|x\|_\infty} \|e_i\|$$

$$\leq \sum_i \|x\|_\infty \|e_i\| = \|x\|_\infty \underbrace{\sum_i \|e_i\|}_{=: c_1} \Rightarrow \|x\| \leq c_1 \|x\|_\infty$$

Second Inequality: $\exists C_2 > 0 \ \forall x \ \|x\|_\infty \leq C_2 \cdot \|x\|$

Let $S := \{x \in \mathbb{R}^n \mid \|x\|_\infty = 1\}$ be the unit sphere w.r.t. $\|\cdot\|_\infty$

Consider $f: S \to \mathbb{R}$, $x \longmapsto \|x\|$

The mapping f is continuous w.r.t. $\|\cdot\|_\infty$:

this follows from the fact that:

$$| f(x) - f(y) | = \big| \|x\| - \|y\| \big| \leq \|x-y\| \leq C_1 \|x-y\|_\infty$$

Lipshictz continuity

The S is closed and bounded, so S is compact (from analysis). Any continuous mapping on a compact set takes its min and max.

define $\tilde{C}_2 := \min \{f(x) \mid x \in S\}$

$$x \in S: \quad \|x\| = \left\| \frac{x}{1} \right\| = \left\| \frac{x}{\|x\|_\infty} \right\| = \frac{\|x\|}{\|x\|_\infty}$$

since $x \in S$, $\|x\|_\infty = 1$

$$\Rightarrow \tilde{C}_2 \leq \frac{\|x\|}{\|x\|_\infty} \Rightarrow \|x\|_\infty \leq \underbrace{\frac{1}{\tilde{C}_2}}_{C_2 := 1/\tilde{C}_2} \|x\|$$

$$\|x\|_\infty \leq C_2 \|x\|$$

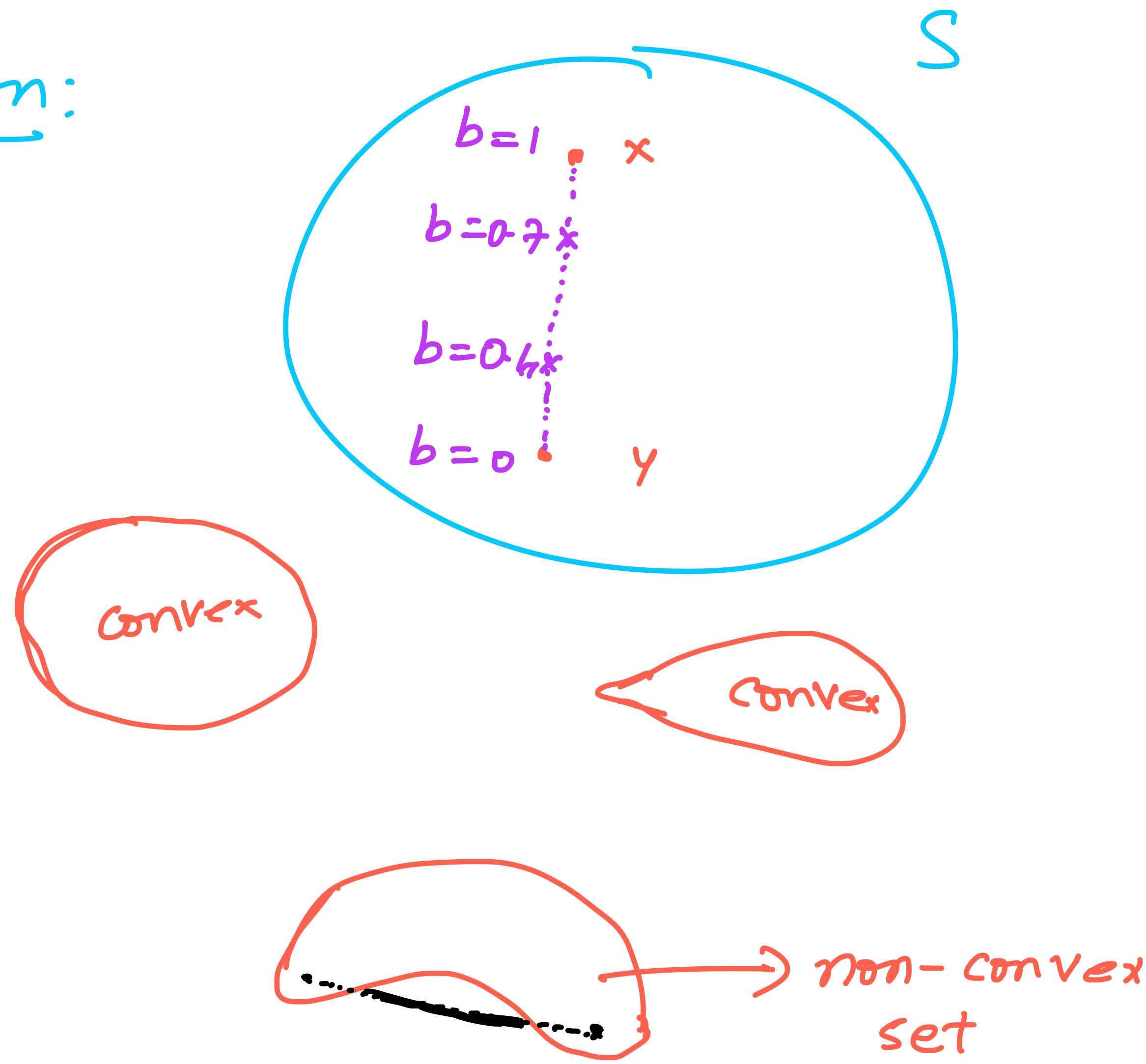
$\blacksquare$

Convex Sets are Unit balls of norms

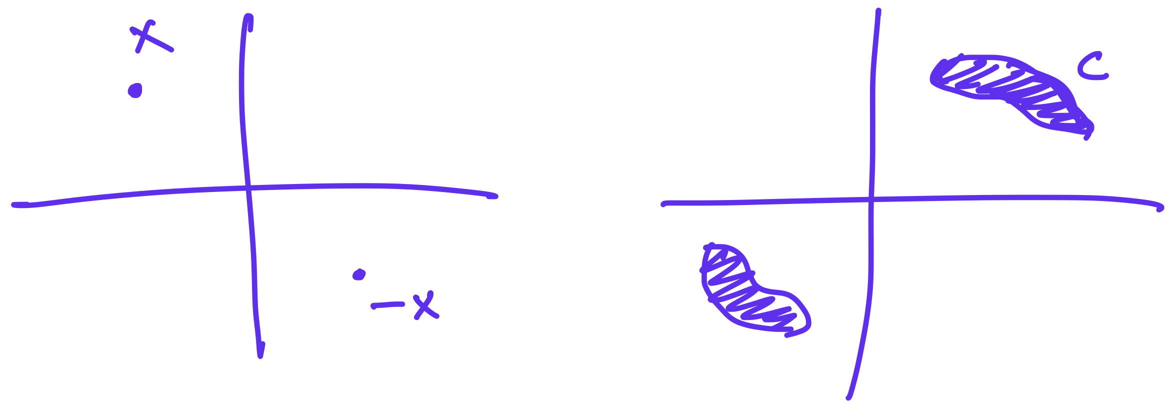
Def: Consider a real vector space, V, $S \subset V$. S is called convex if $\forall\, b$, $0 \leq b \leq 1$ and $\forall\, x, y \in S$,

$$b \cdot x + (1-b) \cdot y \in S$$

Intuition:

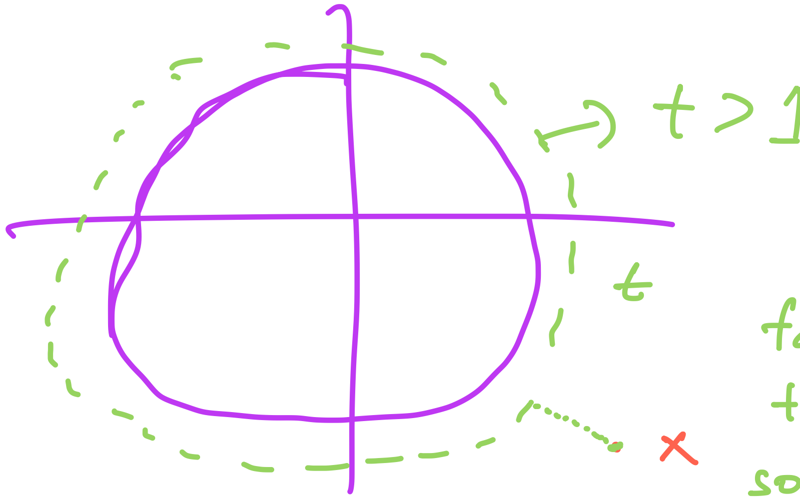


Def: A set $C \subset V$ is called symmetric if $x \in C \Rightarrow -x \in C$



Theorem: (1) Let $C \subset \mathbb{R}^d$ closed, convex, symmetric and has non-empty interior. Define $p(x) := \inf\{t > 0 \mid x \in t \cdot C\}$. Then p is a semi-norm. If C is bounded, then p is a norm, and its unit ball coincides with C. (i.e. $C = \{x \in \mathbb{R}^d \mid p(x) \leq 1\}$

(2) For any norm $\|\cdot\|$ on $\mathbb{R}^d$, the set $C := \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$ is bounded, symmetric, closed, convex and has non-empty interior.



the smallest factor I need to multiply c so that I reach 'x'.

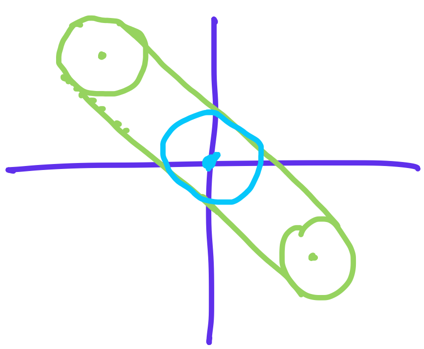
Proof: $\boxed{p(x) \text{ is well defined}}$

Want to prove: given $x \in \mathbb{R}^d$, the set $\{t > 0 \mid x \in t \cdot c\} \neq \emptyset$.

We are going to prove: $\exists\, \varepsilon > 0$ such that

$$B_\varepsilon(0) = \{e \in \mathbb{R}^d \mid \|e\|_2 < \varepsilon\} \in C$$

Intuition:



- By assumption, C has at least one interior point.

$v \in C^\circ \Rightarrow \exists\, \epsilon$ such that

$$B_\varepsilon(v) \in C \Rightarrow v + B_\varepsilon(0) = \{v + e \mid e \in B_\varepsilon(0)\}$$

- By symmetry, $v+e \in C \Rightarrow -(v+e) \in C$
- By convexity, $\frac{1}{2}(v+e)+\frac{1}{2}(-(v+e)) = e \in C$

So $B_\varepsilon(0) \subset \underline{C}$, so the set $\{\underline{t}>0 \mid x \in t \cdot C\}$ is non-empty.

The infimum of $\inf\{t>0 \mid x \in t \cdot C\}$ exists because $\{t>0 \mid x \in t \cdot C\} \subset \mathbb{R}$ has 0 as its lower bound.

(P1) $\boxed{p(0) = 0}$

- have seen: $0 \in C$
- $\forall t>0: \quad 0 \in \underbrace{0 \cdot C}_{0}$
- $\inf\{t \mid 0 \in t \cdot C\} = 0$

$\Rightarrow \quad p(0) = 0$

(P2) $\boxed{p(\alpha x) = |\alpha|\, p(x)}$

- $\forall \alpha > 0$, we have

$$p(\alpha \cdot x) = \inf \{ t > 0 \,/\, \alpha \cdot x \in t \cdot C \}$$

$s := t/\alpha$

$$= \inf \{ \alpha \cdot s > 0 \mid x \in s \cdot C \}$$

$$= \alpha \cdot \underbrace{\inf \{ s > 0 \mid x \in s \cdot C \}}_{p(x)}$$

$$= \alpha \, p(x)$$

$\Rightarrow \quad p(\alpha x) = \alpha \cdot p(x)$

- By symmetry we also get

$$p(-x) = p(x)$$

- Combining the two statements

to say $p(\alpha \cdot x) = |\alpha| \cdot p(x)$

(homogenity)

(P3) $\boxed{\text{Triangle - inequality.}}$

Consider $x, y \in \mathbb{R}^d$, $s, t > 0$ such that.

$$\frac{x}{s} \in C, \frac{y}{t} \in C.$$

<u>Observe:</u> $\frac{s}{s+t} + \frac{t}{s+t} = 1$, Then, by convexity.

$$\frac{s}{s+t} \cdot \underbrace{\frac{x}{s}}_{\in C} + \frac{t}{s+t} \cdot \underbrace{\frac{y}{t}}_{\in C} \in C \Rightarrow \frac{x+y}{s+t} \in C$$

two scalars that sum to 1

$$\Rightarrow \frac{x+y}{u_0} \in S$$

$$\Rightarrow p(x+y) = \inf\{u > 0 \mid x+y \in u \cdot C\} \leq u_0$$

$$\leq s + t$$

$$= p(x) + p(y)$$

s was chosen such that $x \in s \cdot C$

t " " " " $y \in t \cdot C$

Consider a sequence $(s_i)_{i\in\mathbb{N}}$ such that

$x \in s_i \cdot C$ and $s_i \to p(x)$

Similarly $(t_i)_{i\in\mathbb{N}}$ such that $y \in t_i \cdot C$ and

$t_i \to p(y)$

$$\forall i : \quad p(x+y) \leq \underbrace{s_i}_{\to p(x)} + \underbrace{t_i}_{\to p(y)}$$

$$\Rightarrow \quad p(x+y) \leq p(x) + p(y)$$

(P4) $\boxed{p(x) = 0 \Rightarrow x = 0}$

$p(x) = 0 \iff \inf\{t > 0 \mid x \in t \cdot C\} = 0$

$\Rightarrow$ There exists a sequence $(t_k)_{k\in\mathbb{N}}$ s.t.

$t_k \to 0$ and $x \in t_k \cdot C \;\; \forall k.$

Now assume that $x \neq 0$. Then the sequence $\left(\frac{x}{t_k}\right)_{k\in\mathbb{N}}$ is unbounded

$\to$ contradiction since C is bounded.

$\blacksquare$

Normed Function Spaces

Space of continuous functions

Let T be a metric space,

$$\mathcal{C}^b(T) := \left\{ f : T \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is continuous} \\ \text{and bounded} \end{array} \right\}$$

$\exists c \in \mathbb{R}$:
$\forall t \in T : |f(t)| < c$

As norm on $\mathcal{C}^b(T)$ we choose:

$$\|f\|_\infty := \sup_{t \in T} |f(t)|$$

The norm exists since we are in the space of bounded functions, bounded from above.

Then the space $\mathcal{C}^b(T)$ with norm $\|\cdot\|_\infty$ is a Banach space

If $(X, d_{\|\cdot\|})$ is a complete metric space, then the normed space $(X, \|\cdot\|)$ is called a Banach Space.

Proof outline:

(i) need to check vector space axioms

(ii) norm axioms

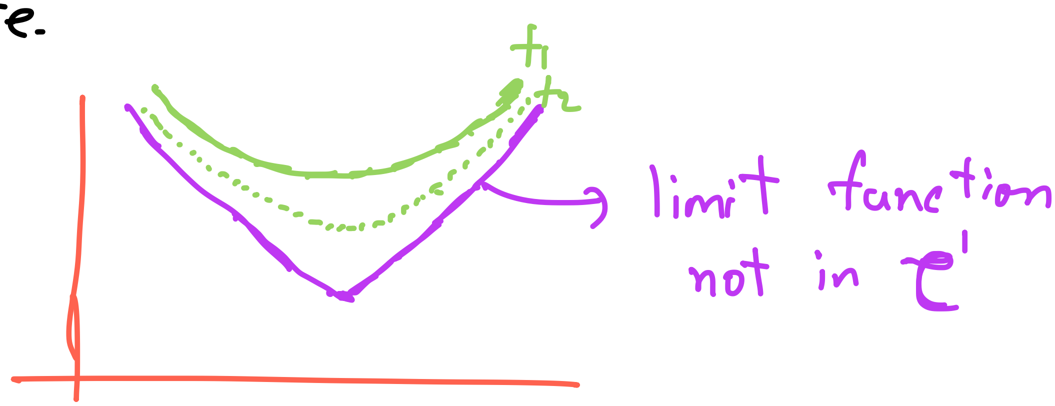
(iii) completeness: follows from the fact that $\|\cdot\|_\infty$ induces uniform convergence.

Space of differentiable functions:

Let $[a,b] \subset \mathbb{R}$, $\mathcal{C}^1([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuously differentiable}\}$

Which norm?

- Consider $\|\cdot\|_\infty$. With this norm, $\mathcal{C}^1$ is not complete.



Is there a better norm. The answer is yes.

Many norms exist, let us consider a few examples.

- Consider $\|f\| := \sup_{t \in [a,b]} \max\{|f(t)|, |f'(t)|\}$

- Consider $\|f\| := \|f\|_\infty + \|f'\|_\infty$

$\mathcal{C}^1([a,b])$ with any of these two norms is a Banach space.