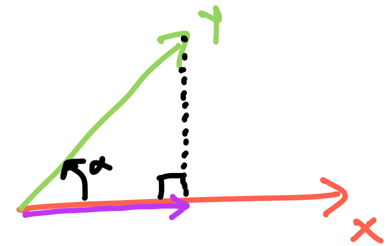
# Inner Product and Hilbert Spaces

- metric $\longrightarrow$ measures distances
- norm $\longrightarrow$ measures distances, lengths
- inner product $\longrightarrow$ measures distances, lengths, angles.

" $\langle x, y \rangle = \|x\| \cdot \|y\| \cos(\alpha)$ "



In ML we use cosine similarity : $\cos(\theta) = \dfrac{\langle x, y\rangle}{\|x\| \, \|y\|}$

Inner Product $\Longleftrightarrow$ scalar product $\Longleftrightarrow$ dot product

**Def:** Consider a vector space $V$. A mapping $\langle \cdot, \cdot \rangle : V \times V \to F$ is called an inner product if

linearity
(P1) $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$
(P2) $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle \quad (\lambda \in F)$

symmetry
(P3) $\langle x, y\rangle = \langle y, x\rangle$ (if $F$ is $\mathbb{R}$)
$\langle x, y\rangle = \overline{\langle y, x\rangle}$ (if $F$ is $\mathbb{C}$)
complex conjugate

positive definite
(P4) $\langle x, x\rangle \geq 0$
(P5) $\langle x, x\rangle = 0 \iff x = 0$

Examples: • Euclidean inner product on $\mathbb{R}^n$:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \qquad x=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

• On $\mathbb{C}$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

• $\mathcal{C}([a,b])$ : $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$

is an inner product (but space would not be complete)

**Def:** A vector space with a norm is called a <u>normed space</u>. If a normed space is complete (all Cauchy sequences converge), then V is called a <u>Banach Space</u>. A vector space with an inner product is called a <u>pre-Hilbert-space</u>. If it is additionally complete, then V is called a <u>Hilbert space</u>.

$$\boxed{\begin{array}{l} \text{inner product} \Rightarrow \text{norm} \\ \text{inner product} \not\Leftarrow \text{norm} \end{array}}$$

Consider a vector space with an inner product $\langle \cdot, \cdot \rangle$. Define $\|\cdot\| : V \to \mathbb{R}$ as $\|x\| := \sqrt{\langle x, x \rangle}$. Then $\|\cdot\|$ is a norm on $V$, the norm is induced by $\langle \cdot, \cdot \rangle$.

In general, the other way does not work.

$$\boxed{\text{norm} \underset{\not\Leftarrow}{\Rightarrow} \text{metric}}$$

Consider a vector space $V$ with norm $\|\cdot\|$. Then $d : V \times V \to \mathbb{R}$, $d(x, y) := \|x - y\|$ is a metric on $V$, the metric is induced by the norm.

In general, the other direction does not work.

$$\boxed{\text{inner product} \underset{\not\Leftarrow}{\Rightarrow} \text{norm} \underset{\not\Leftarrow}{\Rightarrow} \text{metric}}$$

# Orthogonal Basis and Projections

**Def:** Consider a pre-Hilbert-space $V$. Two vectors $v_1, v_2 \in V$ are called orthogonal if $\langle v_1, v_2 \rangle = 0$.

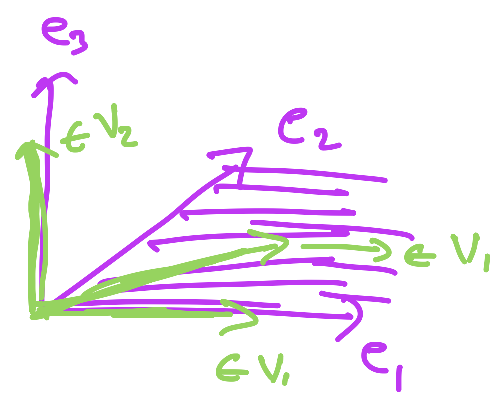
Notation: $v_1 \perp v_2$

Two sets $V_1, V_2 \subset V$ are called orthogonal if $\forall v_1 \in V_1, v_2 \in V_2 : \langle v_1, v_2 \rangle = 0$

$V_1 = \text{span}\{e_1, e_2\}$

$V_3 = \text{span}\{e_3\}$

$V_1 \perp V_3$



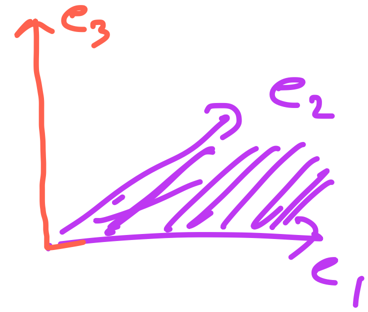

Vectors are called orthonormal if additionally the two vectors have norm of 1:

- $\langle v_1, v_2 \rangle = 0$
- $\|v_1\| = 1, \quad \|v_2\| = 1$

A set of vectors $v_1, v_2, \dots v_n$ is called orthonormal if any two vectors are orthonormal.

For a set $S \subseteq V$ we define its orthogonal complement $S^\perp$ as follows:

$$S^\perp := \{ v \in V \mid v \perp s,\ \forall s \in S \}$$




$V_1 = \operatorname{span}\{e_1, e_2\}$

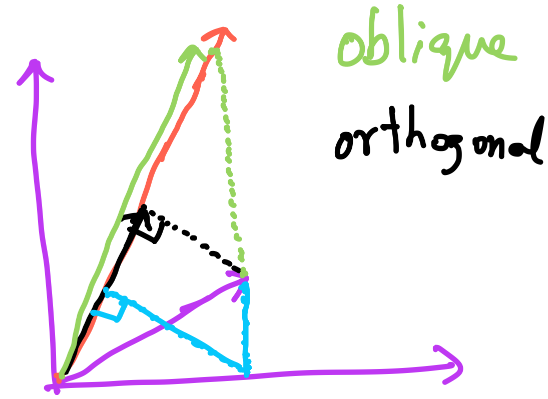
$V_2 = \operatorname{span}\{e_3\}$

$V_1^\perp = V_2, \quad V_2^\perp = V_1$

## Orthogonal Projections

**Def:** $A \in \mathcal{L}(V)$ is called a projection if

$$A^2 = A.$$




**Theorem & Def:** Let $U$ be a finite-dim subspace of a pre-Hilbert-space $H$. Then there exists a linear projection $P_U : H \to U$, and $\ker(P_U) = U^\perp$. $P_U$ is then called the orthogonal projection of $H$ on $U$.

**Construction:** Let $v_1, \dots, v_n$ be an orthogonal basis of $U$. Define $P_U : V \to U$ by

$$P_u(w) = \sum_{i=1}^{n} \frac{\langle w, v_i \rangle}{\|v_i\|_2^2} v_i$$

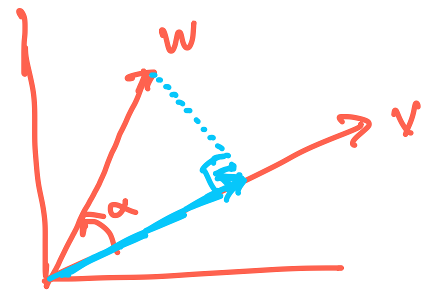
**Intuition:**

$\|w\| = 1$

If $\|v\| = 1$, then

$$\|P_v(w)\| = |\langle v, w \rangle|$$

$$= |\cos \alpha|$$




In particular $\langle v, w \rangle = \cos \alpha$

**Remark:** In an orthonormal basis $u_1 \dots u_n$ the representation of a vector $v$ is given by

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$$

## Gram-Schmidt orthogonalization

It is a procedure that takes any basis $v_1 \dots v_n$ of a finite-dim vector space and transforms it into another basis $u_1 \dots u_n$ that is orthonormal.

Intuition: iterative procedure

Step 1: $$u_1 = \frac{v_1}{\|v_1\|}$$

$$U_1 = \text{span}\{u_1\}$$

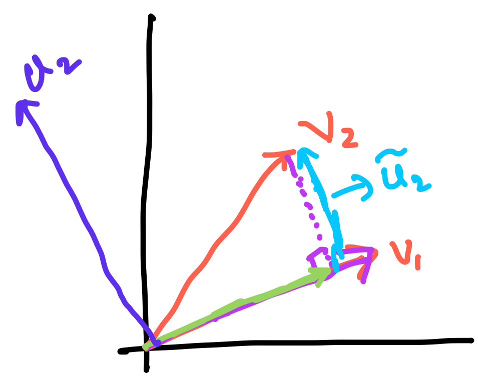
Step K: Assume that we already have $u_1, u_2 \ldots u_{k-1}$.

- Project $v_k$ on $U_{k-1}$ and keep "the rest"

$$\tilde{u}_k = v_k - P_{U_{k-1}}(v_k)$$

- Renormalize:

$$u_k = \frac{\tilde{u}_k}{\|\tilde{u}_k\|}$$




In practice use Householder reflections for a numerically stable orthogonalization.

# Orthogonal Matrices

**Def:** Let $Q \in \mathbb{R}^{n \times n}$ be a matrix with orthonormal column vectors (w.r.t. Euclidean inner product). Then $Q$ is called an orthogonal matrix.

If $Q \in \mathbb{C}^{n \times n}$ and the columns are orthonormal (w.r.t. the standard inner product on $\mathbb{C}$), then it is called unitary.

Examples:

- Identity: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection about x-axis
- Permutation: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Rotation: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

- Rotation in $\mathbb{R}^3$
  - rotate about one of the axes:

$$R_{\theta,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

  - General rotation: can be written as a product of "elementary" rotation

<u>Properties</u> of orthogonal matrix $Q$:

- columns are orthogonal $\Longleftrightarrow$ rows are orthogonal
- $Q$ is always invertible, and $Q^{-1} = Q^T$
- $Q$ realizes an isometry : $\forall v \in V : \|Qv\| = \|v\|$

  ↳ keeps lengths intact
- $Q$ preserves angles: $\langle Qu, Qv \rangle = \langle u, v \rangle$ $\forall u, v \in V$
- $|\det Q| = 1$

The respective properties also hold for unitary matrices $U$. $(U^{-1} = \bar{U}^T)$

**Theorem:** Let $S \in \mathcal{L}(V)$ for a real vector space $V$. Then the following are equivalent:

(a) $S$ is an isometry: $\|Sv\| = \|v\|$, $v \in V$.

(b) There exists an orthonormal basis of $V$ such that the matrix of $S$ has the following form:

$$M = \begin{pmatrix} \square & & & & 0 \\ & \square & & & \\ & & \square & & \\ 0 & & & \square & \\ & & & & \ddots \ \square \end{pmatrix}$$

where each of the little block.

- either a $1\times1$ matrix (one real number) with a value $+1$ or $-1$.

- or a $2\times2$ rotation matrix.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$