

# Symmetric Matrices

Def: A matrix  $A \in \mathbb{R}^{n \times n}$  is called symmetric if  $A = A^T$

A matrix  $A \in \mathbb{C}^{n \times n}$  is called Hermitian if  $A = \overline{A}^T$

Proposition: Let  $A \in \mathbb{C}^{n \times n}$  be Hermitian.

Then all eigenvalues of  $A$  are real-valued. Eigenvectors that correspond to distinct eigenvalues are orthogonal.

Proof:  $\lambda$  eigenvalue of  $A$  with eigenvector  $x$ . Then.

$$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle Ax, x \rangle =$$

$$\langle x, Ax \rangle = \langle x, \lambda x \rangle = \overline{\lambda} \langle x, x \rangle$$

$$\Rightarrow \underline{\lambda = \overline{\lambda}} \in \mathbb{R} \quad (\text{unless } x = 0 \text{ vector})$$

$\hookrightarrow \lambda$  has to be real.

$(\lambda_1, x_1), (\lambda_2, x_2)$  are eigenvalue - eigenvector pairs of  $A$ . Then

$$\begin{aligned}\lambda_1 \langle x_1, x_2 \rangle &= \langle \lambda_1 x_1, x_2 \rangle = \langle Ax_1, x_2 \rangle = \langle x_1, Ax_2 \rangle \\ &= \langle x_1, \lambda_2 x_2 \rangle = \bar{\lambda}_2 \langle x_1, x_2 \rangle\end{aligned}$$

$$\hookrightarrow \bar{\lambda}_2 = \lambda_2$$

$$0 = \lambda_1 \langle x_1, x_2 \rangle - \lambda_2 \langle x_1, x_2 \rangle$$

$$0 = (\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle$$

$$\Rightarrow \text{either } \lambda_1 = \lambda_2$$

$$\text{or if } \lambda_1 \neq \lambda_2, \text{ then } \langle x_1, x_2 \rangle = 0$$

$$\Rightarrow x_1 \perp x_2$$



Def: An operator  $T \in \mathcal{L}(V)$  on a pre-Hilbert space  $V$  is called self-adjoint if

$$\langle Tv, w \rangle = \langle v, Tw \rangle.$$

Sometimes it is called a Hermitian operator (on  $\mathbb{C}^n$ )  
Symmetric operator (on  $\mathbb{R}^n$ )

Remark: Over  $\mathbb{C}^n$ , self-adjoint operators are represented by Hermitian matrices. On  $\mathbb{R}^n$ , self-adjoint operator is represented by a symmetric matrix.

Proposition:  $T \in \mathcal{L}(V)$  self-adjoint. Then  $T$  has at least one eigenvalue, and it is real-valued. (holds both on  $\mathbb{C}^n$  and  $\mathbb{R}^n$ ).

Proof (sketch):  $n := \dim V$ . Chose  $v \neq 0$ , and consider,  $v, Tv, T^2v, \dots, T^n v$

These vectors have to be linearly dependent ( $n+1$  vectors,  $\dim = n$ ).

There exists  $a_0, a_1, \dots, a_n$ :

$$a_0 v + a_1 Tv + \dots + a_n T^n v = 0.$$

Consider polynomial with these coefficients:

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$$

$$= c \underbrace{(x^2 + b_1 x + c_1) \dots (x^2 + b_m x + c_m)}_{\text{quadratic terms}} \underbrace{(x - \lambda_1) \dots (x - \lambda_m)}_{\text{linear terms}}$$

Replace the  $x$  by  $T$ :

$$0 = (a_0 + a_1 T + \dots + a_n T^n)v = \left( c(\underbrace{\dots}_{\text{quadratic}}) \underbrace{(\dots)}_{\text{linear}} \right)v$$

Now can show: the quadratic terms are invertible, and we are left with (at least one) linear factor:

$$0 = (T - \lambda_1 I) \dots (T - \lambda_n I)v$$

There needs to exist at least one 'i' such that  $(T - \lambda_i I)$  is not invertible

$$\text{So. } (T - \lambda_i I)v = 0 \Rightarrow Tv = \lambda_i v$$

$\Rightarrow \lambda_i$  is a eigenvalue of  $T$ .



# Spectral Theorem for Symmetric / Hermitian Matrices

Theorem: A symmetric matrix  $A \in \mathbb{R}^{n \times n}$  is orthogonally diagonalizable: there exists an orthogonal matrix  $Q \in \mathbb{R}^{n \times n}$  and a diagonal matrix  $D \in \mathbb{R}^{n \times n}$  s.t.

$$A = Q D Q^T$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= \sum_{i=1}^n \lambda_i \underbrace{q_i q_i^T}$$

$\hookrightarrow$  rank-1 matrices

Theorem: A Hermitian matrix  $A \in \mathbb{C}^{n \times n}$  is unitarily diagonalizable: there exists a unitary matrix  $U$  and a diagonal matrix  $D$  s.t.

$$A = U D \bar{U}^T$$

the entries of  $D$  are real-valued.

# Positive Definite Matrices

Def: A matrix  $A \in \mathbb{R}^{n \times n}$  is called a positive definite (PD) if  $\forall x \in \mathbb{R}^n$ ,  $x \neq 0$ ,  $x^T A x > 0$

For positive semi-definite (PSD)  $\forall x \in \mathbb{R}^n$ ,  $x \neq 0$ ,  $x^T A x \geq 0$

Def: A matrix  $A \in \mathbb{C}^{n \times n}$  is called a Gram matrix if there exists a set of vectors  $v_1, \dots, v_n \in \mathbb{C}^n$  such that  $a_{ij} = \langle v_i, v_j \rangle$ . Note: Gram matrices are Hermitian (similarly on  $\mathbb{R}^{n \times n}$ , then Gram matrices are symmetric).

$$G = V^T V, \quad V = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$$
$$C = V V^T$$

⚠ Over  $\mathbb{C}$ , we have that PD  $\Rightarrow$  self adjoint.

Over  $\mathbb{R}$ , this is not true!

$\Rightarrow$  there are matrices which are PD but not symmetric.

Example:  $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$x^T A x = x_1^2 + x_2^2 > 0$$

$\rightarrow$  So  $A$  is PD but not symmetric

$\rightarrow$  Over  $\mathbb{C}$ , the same matrix is not

PD since  $x_1^2 + x_2^2$  can be negative!

Theorem:  $A \in \mathbb{C}^{n \times n}$  Hermitian. Then equivalent:

(i)  $A$  is PSD (PD)

(ii) All eigenvalues of  $A$  are  $\geq 0$  ( $> 0$ )

(iii) The mapping  $\langle \cdot, \cdot \rangle_A : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$  with

$$\langle x, y \rangle_A := \bar{y}^T A x$$

satisfies all properties of an inner product

except one: if  $\langle x, x \rangle_A = 0$ , this does

not imply  $x = 0$

(This mapping is an inner product)

(iv)  $A$  is a Gram matrix of  $n$  vectors which are not necessarily linearly independent

(which are linearly independent).

$$a_{ij} = \langle x_i, x_j \rangle$$



## Roots of PSD matrices

Theorem: Let  $A \in \mathbb{R}^{n \times n}$  be symmetric, PSD. Then there exists a matrix  $B \in \mathbb{R}^{n \times n}$ ,  $B$  is PSD such that  $A = B^2$ . Sometimes  $B$  is called the square root of  $A$ ,

$$\underline{B = (A)^{1/2}}$$

Proof: Spectral theorem  $\Rightarrow$

$$A = U D U^T, \quad D \text{ diagonal}$$

• PSD  $\Rightarrow$  eigenvalues  $\geq 0$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \lambda_i \geq 0$$

Define  $\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$  and set

$$B := U \sqrt{D} U^T.$$



# Variational Characterization of Eigenvalues

Def: Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix.

$$R_A : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$$

$$x \mapsto \frac{x^T A x}{x^T x}$$

is called the Rayleigh coefficient.

Proposition: Let  $A$  be symmetric, let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  be the eigenvalues and  $v_1, \dots, v_n$  the eigenvectors of  $A$ .

$$\text{Then : } \min_{x \in \mathbb{R}^n} R_A(x) = \min_{\|x\|=1} x^T A x$$

$$= \lambda_1, \text{ attained at } x = v_1$$

$$\max_{x \in \mathbb{R}^n} R_A(x) = \max_{\|x\|=1} x^T A x$$

$$= \lambda_n, \text{ attained at } x = v_n$$

Intuition: Assume  $A$  is expressed in terms of the basis  $v_1, \dots, v_n$

$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ . Let  $y$  be a vector, also represented in the same basis.

$$y = y_1 v_1 + y_2 v_2 + \dots + y_n v_n$$

$$y^T A y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

Among the vectors  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

the smallest result of  $y^T A y$  would be given by the vector  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ , and the value would be  $\lambda_1$ .   
  $\rightarrow v_1$

## More general proof (sketch):

Assume we start with the standard basis. Let  $Q = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \dots & \\ & & & 1 \end{pmatrix}$  be the basis transform

$$\lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$