CSE 840: Computational Foundations of Artificial Intelligence	Oct. 4, 2023
Lecture 10	
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1 Calculus

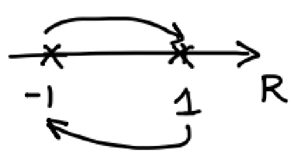
Limits: $\lim_{b\to a} \frac{f(b)-f(a)}{b-a}$

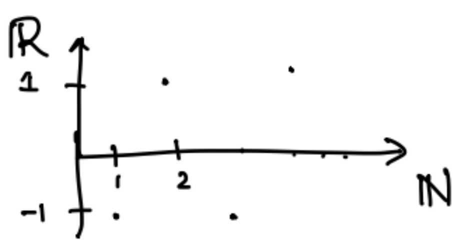
Derivatives: -> optimization problems

Integrals: -> expectation

2 Sequences

Examples: $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, ...)$





(b) $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, ...)$

($\lim_{n\to\infty} a_n = 0$)

(c) $(a_n)_{n\in\mathbb{N}} = (2^b)_{n\in\mathbb{N}} = (2, 4, 8, 16, ...)$

Definition 1 *A sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent to $a \in \mathbb{R}$ if $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : |a_n - a| < \epsilon$*

If there is no such $a \in \mathbb{R}$, then sequence diverges.

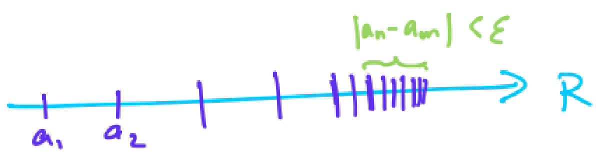
Definition 2 *A sequence $(a_n)_{n\in\mathbb{N}}$ is called bounded if $\exists c \in \mathbb{R} \forall n \in \mathbb{N} : |a_n| \leqq c$*

otherwise, the sequence is unbounded.

Fact 3 *$(a_n)_{n\forall\mathbb{N}}$ convergent => $(a_n)_{n\forall\mathbb{N}}$ bounded.*

$(a_n)_{n\forall\mathbb{N}}$ => There is only one limit ($\lim_{n\to\infty} = a$) $a \in \mathbb{R}$

Definition 4 *If $\forall > 0 \exists N \in \mathbb{N} \forall n, m \geqq N : |a_n - a_m < \epsilon$ then $(a)_{n\in\mathbb{N}}$ is called a Cauchy Sequence.*



Fact 5 *For a sequence of real numbers: Cauchy Sequence <==> Convergent Sequence*

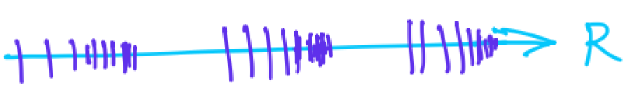
Proposition 6 *If $(a_n)_{n\forall\mathbb{N}}$ is monotonically decreasing ($a_{n+1} \leqq a_n \forall n$) and bounded from below (the set $(a_n)_{n\in\mathbb{N}}$ has a lower bound), then $(a_n)_{n\in\mathbb{N}}$ is convergent.*

Example subsequence: $(a_n)_{n\in\mathbb{N}} = (-1)^n$

subsequence: $(a_n)_{n\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}} = (1, 1, ...1) -> 1$

subsequence: $(a_n)_{n\in\mathbb{N}} = (a_{2k+1})_{k\in\mathbb{N}} = (-1, -1, ...) -> -1$

Definition 7 *$a \in \mathbb{R}$ is called an accumulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subsequence $(a_{nk})_{k\in\mathbb{N}}$ with* $(\lim_{k\to\infty} a_{nk} = a)$.



(cluster point, accumulation point, limit paint, partial limit ...)

Theorem 8 *Bolzano-Weierstrass theorem.*

$(a_n)_{n\in\mathbb{N}}$ bounded => $(a_n)_{n\in\mathbb{N}}$ has an accumulation value (has a convergent subsequence)

Observation 9

- *A sequence can have many accumulation points (or no accumulation point)*
- *Even if the sequence has just one accumulation point, it is not necessarily a Cauchy sequence.*
- *If $(a_n)_{n\in\mathbb{N}}$ converges to a, then a is the only accumulation point and the sequence is Cauchy sequence.*

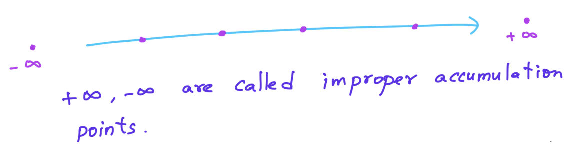
Example: $(a_n)_{n\in\mathbb{N}} = \dfrac{1}{n}$ on $(0,1]$

$(a_n)_{n\in\mathbb{N}}$ is Cauchy, but does not converge on $(0,1]$. It does converge to 0 on $[0,1]$.

Max, Sup, Min, Inf assume we are on $\mathbb{R}$ (or more generally, on a space that has a total ordering). Let $U \subset \mathbb{R}$ to be a subset.

- $x \in \mathbb{R}$ is called maximum element of U if $x \in U$ and $\forall u \in U$: $u \leqq x$. (1 is max $[0,1]$, $(0,1)$ has no max)
- x is called an upper bound of U if $\forall u \in U : u \leqq x$ (5 is an upper bound of $(0,1)$ or $[0,1]$)
- x is called a supremum of U if it is the smallest upper bound. (1 is the supremum of $(0,1)$)

Analogously define minimum, lower bound and infimum. A give sequence $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values ∞, $-\infty$ are called improper accumulation points.



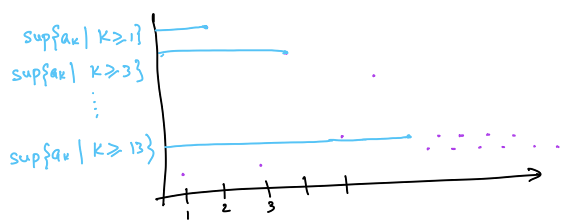
Definition 10 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. An element $A \in \mathbb{R} \cup (-\infty, +\infty)$ is called:*

- *limit superior of $(a_n)_{n\in\mathbb{N}}$ if a is the largest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$*

$$\text{write } a = \limsup_{n\to\infty} a_n$$

- *limit inferior of* $(a_n)_{n\in\mathbb{N}}$ *if a is the smallest (improper) accumulation value of* $(a_n)_{n\in\mathbb{N}}$

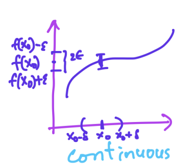
$$\text{write } a = \liminf_{n\to\infty} a_n$$

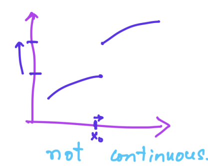


$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$

2.1 Continuity

Definition 11 *A function* $f : (x,d) \to (y,d)$ *between two metric spaces* (X,d) *and* (Y,d) *is called continuous at* $X_0 \in X$ *if* $\forall \epsilon > 0, \exists \delta > 0, \forall x \in X : d(X, X_0) < \delta \implies d(f(X), f(X_0)) < \epsilon$.





Definition 12 *f:* $X \to Y$ *is called continuous at* X_0 *if for every sequence* $(x_n)_{n\in\mathbb{N}} \subset X$

we have: $X_n \to X_0 \implies f(X_n) \to f(X_0)$

A function f: $X \to Y$ *is called* *continuous* *if it is continuous for every* $X_0 \in X$:
$\forall x_0 \in X, \forall \epsilon > 0, \exists \delta > 0, \forall x \in X : d(X, X_0) < \delta$

$$\implies d(f(X), f(X_0)) < \epsilon$$

A function f: $X \to$ *is called* *Lipschitz continuous* *with Lipschitz constant L if*

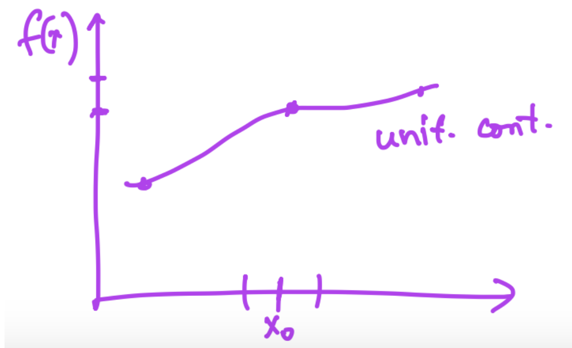
$$\forall x, y \in X : d(f(x), f(y) \leq L \times d(x,y)$$

Intuition: "bounded derivative"

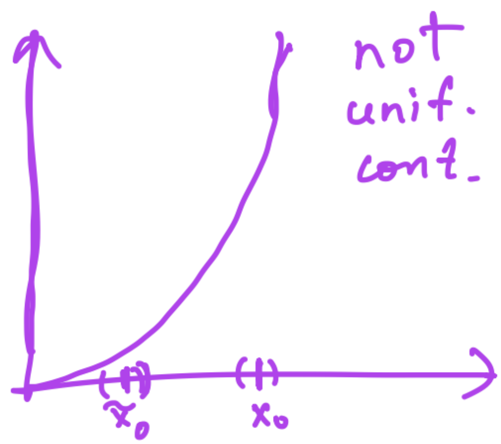
A function f: $X \to Y$ *is called* *uniformly continuous* *if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_0 \in X : d(X, X_0) < \delta$$

$$\implies d(f(X), f(X_0)) < \epsilon$$



Given ϵ, I can choose δ that works for all X_0
Intuition: bounded derivative



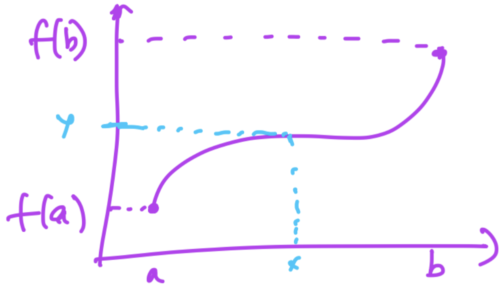
Cannot choose δ to be the same for all X_0
Intuition: unbounded derivative

3 Important theorems for continuous Function

Intermediate Value Theorem:

If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains all values between $f(a) \& f(b)$:

$$\forall y \in [f(a), f(b)], \exists x \in [a, b] : f(x) = y$$



<u>Application</u>: If you want to find X with $f(X) = 0$:

find a with $f(a) < 0$,
find b with $f(b) > 0$
then there must exist $x \in [a, b]$ with $f(x) = 0$

Invertible Functions:

$D \subset R, f : D \to \mathbb{R}$ continuous, strictly monotone ($a < b \Rightarrow f(a) < f(b)$).

Then f is invertible and the inverse is continuous as well.

- invertible follows from monotonicity.

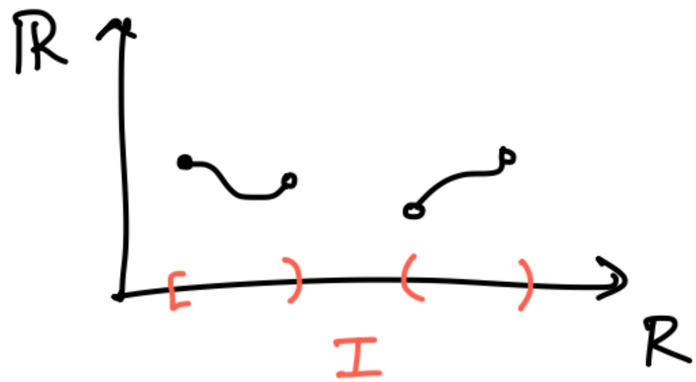
- Continuity of the inverse follows directly from continuity of f.

A function f between two metric spaces $(x, d), (y, d)$ is continuous iff pre-images of the open sets are open:

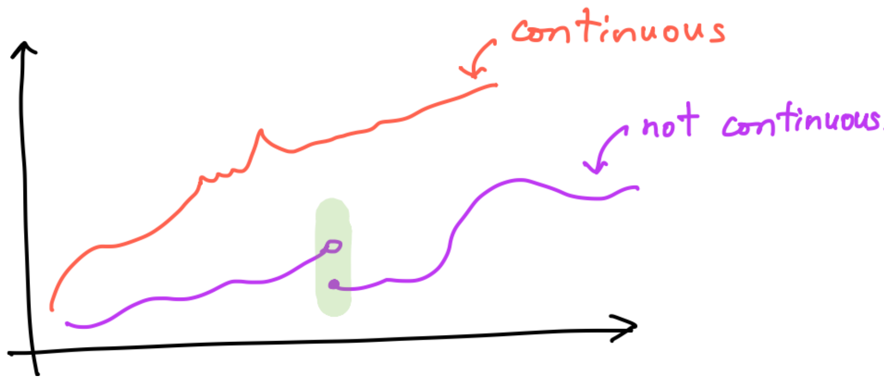
$B \subset y$ open (in y) $\Rightarrow f^{-1}(B) := x \in X | f(x) \in B$ open (in X).

4 Sequences of Functions

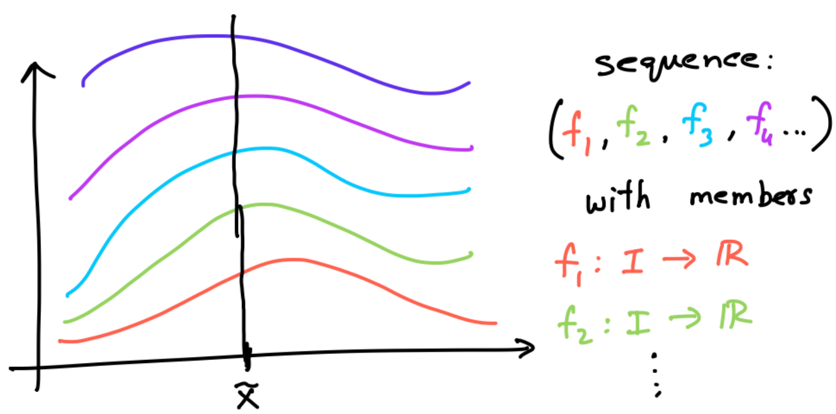
<u>Function:</u> $f : I \to \mathbb{R} (I \subseteq \mathbb{R})$



Continuous functions : $f : \mathbb{R} \to \mathbb{R}$



Idea: small changes on x-axis → small changes on y-axis.



For any fixed $\tilde{x} \in I$, we can get an ordinary sequence of real-numbers.

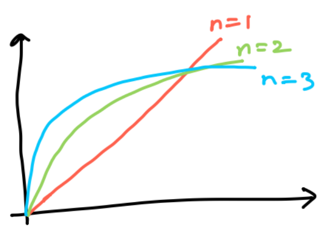
$$\left(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), f_4(\tilde{x}), \ldots\right)$$

Def: Consider functions : $f_n : I \to \mathbb{R}, I \subseteq \mathbb{R}$

We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges prointwise to $f : I \to \mathbb{R}$ if

$$\forall x \in I : f_n(x) \to f(x)$$
$$y_n : f_n(x), y = f(x)$$
$$y_n \to y$$

Example: $f_n, f : [0,1] \to \mathbb{R}, f_n(x) = x^{1/n}$



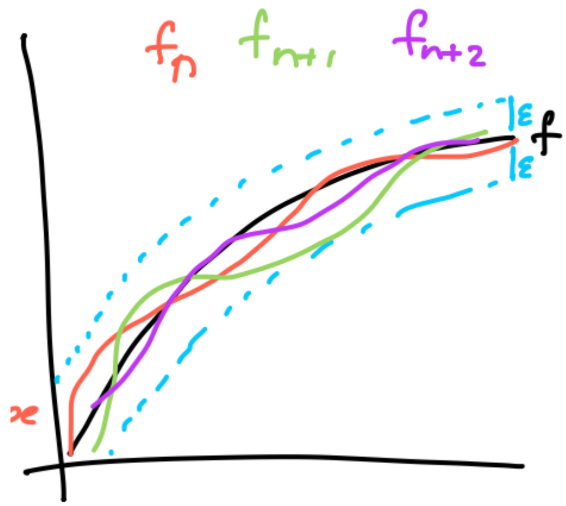
$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & \text{otherwise} \end{cases}$$

⚠ $f_n \to f$ pointwise, all f_n continuous, this doesn't imply that f is continuous.

Def: ${f_n}_{n\in\mathbb{N}}$ converges to f uniformly if

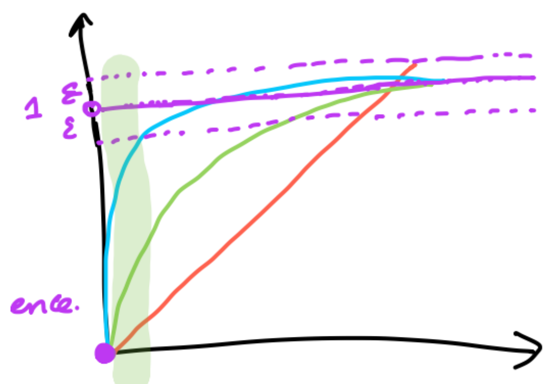
$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad \forall x \in I : |f_n(x) - f(x)| < \epsilon$

Intuition:
Uniform Convergence: given $\epsilon, \exists N$ such that all $f_n \quad n > N$ are contained within $\epsilon-$ tube



Close to zero, there will always be points x close to zero such that the $f_n(x)$ are not yet in $\epsilon-$ tube.

$\Rightarrow$ Not uniformly convergence



Alternative Definition: $f_n \to f$ uniformly iff $||f_n - f||_\infty \to 0$

Theorem: (uniform convergence preserves continuity)

$f_n, f : I \to \mathbb{R}, \quad I \subset \mathbb{R}$, all f_n are continuous, $f_n \to f$ uniformly. Then f is continuous.