CSE 840: Computational Foundations of Artificial Intelligence October 9, 2023

Differentiation, Riemann Integral, Fundamental Theorem of Calculus

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1 Derivatives (one-dimensional case)

Definition 1 *$U \subseteq \mathbb{R}$ an interval, $f : U \to \mathbb{R}$. The function is called differentiable at $a \in U$ if*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. We often write $f' = \frac{df}{dx}$.

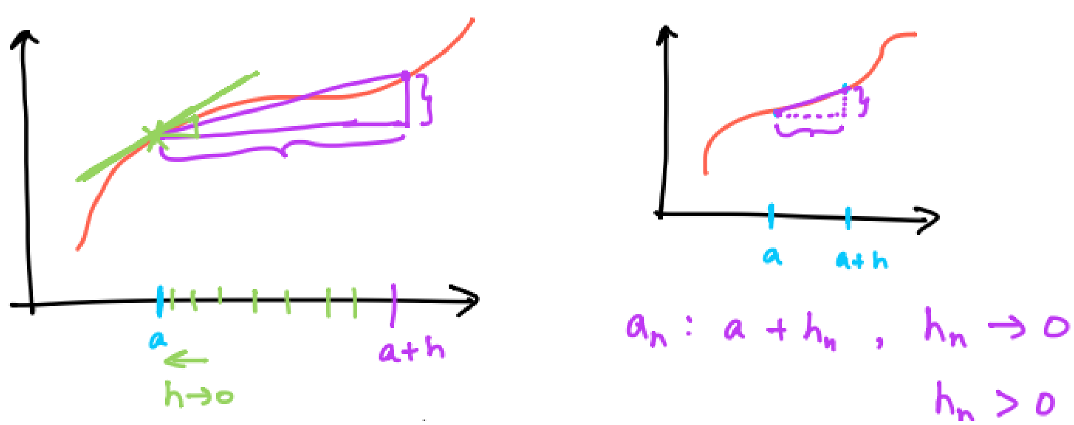

Figure 1: Derivative of a function f as the slope at a

Some intuitions of the definition of a derivative

- The derivative is the slope of a function at a point a
- The derivative is the slope of the linear approximation of a function at a. That is $f(x) = f(a) + (x - a)b$, where b is the slope (derivative at a).

Definition 2 *A function is called differentiable if it has a derivative for all $a \in U$.*

Definition 3 *A function is called continuously differentiable if it is differentiable and the function $f' : U \to \mathbb{R}$, $a \longmapsto f'(a)$ is continuous.*

1.1 Higher Derivatives

We can repeat the process of taking derivatives:

$$f' = \frac{df}{dx}, f'' = \frac{df'}{dx}$$

Notation: $f^{(n)}$ denotes the n-th derivative of f (if it exists)

1.2 Important Theorems

Theorem 4 *(Differentiable $\implies$ Continuous) Let f be differentiable at a. Then there exists a constant c_a such that on a small ball around a we have $|f(x) - f(a)| \leq c_a|x - a|$. In particular, f is continuous at a.*

Theorem 5 *(Intermediate Value Theorem for Derivatives) Let $f \in \mathscr{C}^1([a, b])$ (i.e. functions on $[a, b]$ that are once continuously differentiable), then there exists $\zeta \in [a, b]$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(\zeta)$$

See Figure 2

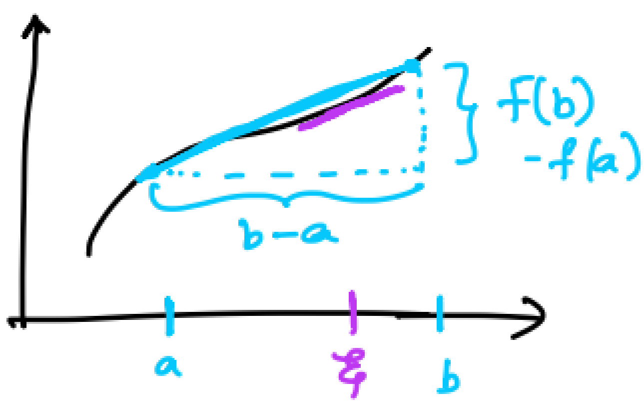

Figure 2: Intermediate Value Theorem for Derivatives: There exists a point on the interval with derivative equal to the slope across the interval

Theorem 6 *(Exchanging limits and derivatives) $f_n : [a, b] \to \mathbb{R}$, $f_n \in \mathscr{C}^1([a, b])$. If the limit $f(x) := \lim_{n\to\infty} f_n(x)$ exists $\forall x \in [a, b]$ and the derivatives f' converge uniformly, then f is continuously differentiable and we have.*

$$f'(x) = (\lim_{n\to\infty})'(x) = (\lim(f_n'))(x)$$

i.e. first take the limit of f_n getting f_1, then finding the derivative is the same as fist finding f_n', then taking the limit.

2 Reimann Integration

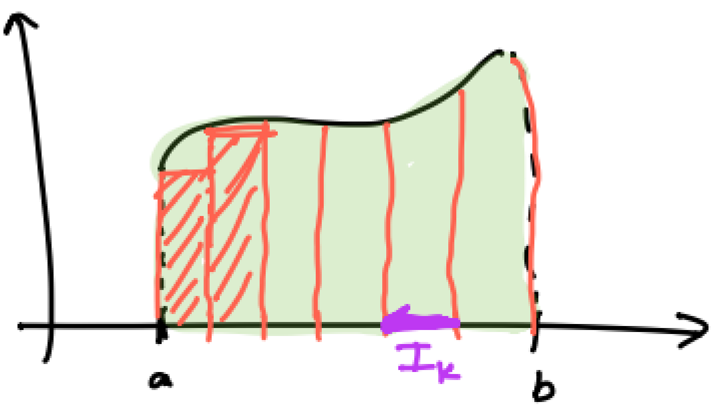

Figure 3: The integral is the area under a curve.

Consider a function $f : [a, b] \to \mathbb{R}$, assume that f is bounded ($\exists \ell, u \in \mathbb{R} \forall x \in [a, b] : \ell \leq f(x) \leq u$). Consider $x_0, x_1, \ldots x_n$ with $a = x_0 < x_1 < \cdots < x_n = b$. These points introduce a partition of $[a, b]$ into n intervals. In particular,

$$I_k := [x_{k-1}, x_k]$$

.

Notice that we can draw more than one rectangle for each of these partitions (Figure 4). In particular we define the heights:

$$m_k := \inf(f(I_k))$$
$$M_k := \sup(f(I_k))$$

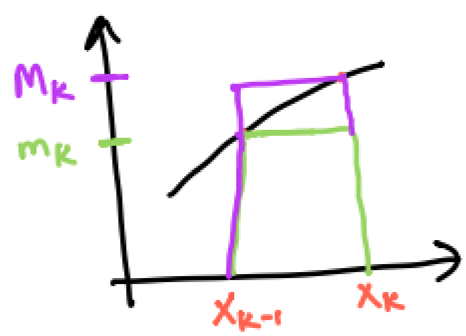

Figure 4: Rectangle can be drawn using either the min or max of the function on the interval as its height

Definition 7 *For a function f and set of partitions $\{x_0, \ldots x_n\}$, the <u>lower sum</u> is:*

$$s(f, \{x_0 \ldots x_n\}) = \sum_{k=1}^{n} |I_k| \cdot m_k$$

where $|I_k|$ *is the length of* $I_k = x_k - x_{k+1}$

Definition 8 *For a function* f *and set of partitions* $\{x_0, \ldots x_n\}$, *the* *upper sum* *is:*

$$S(f, \{x_0 \ldots x_n\}) = \sum_{k=1}^{n} |I_k| \cdot M_k$$

where $|I_k|$ *is the length of* $I_k = x_k - x_{k+1}$

Now we define

$$J_* := \sup_{\text{partitions}} (s(f, \text{partition}))$$

$$J^* := \inf_{\text{partitions}} (S(f, \text{partition}))$$

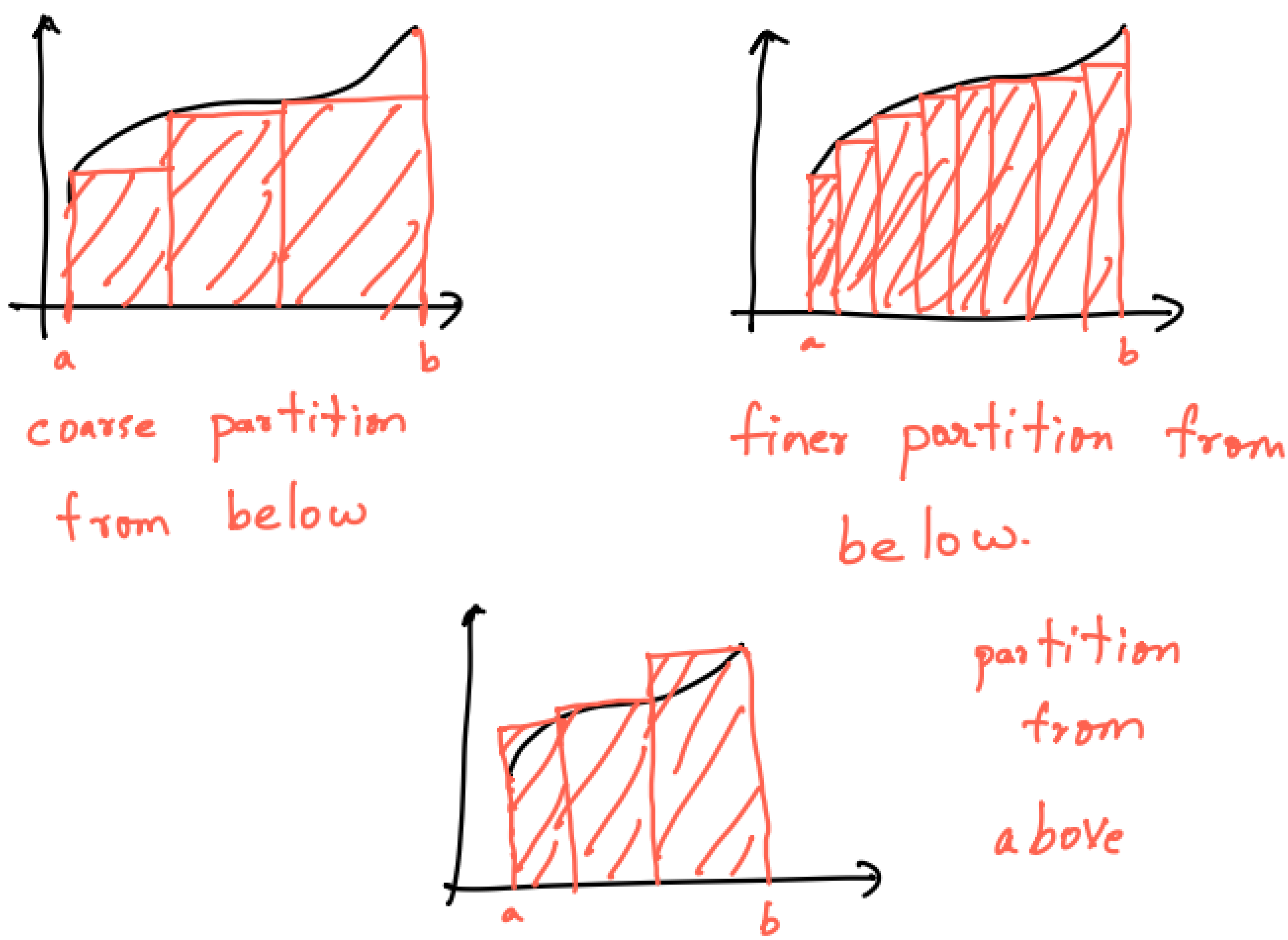

Figure 5: Three different partitions of the same function

Definition 9 *We call a function f Riemann-Integrable if $J_* = J^*$. We then denote*

$$J_* = J^* := \int_a^b f(t)\,dt$$

Theorem 10 • *$f : [a,b] \to \mathbb{R}$ monotone $\implies$ integrable. (i.e. $x_1 < x_2 \implies f(x_1) < f(x_2)$)*

- *$f : [a,b] \to \mathbb{R}$ continuous $\implies$ integrable. (This is true even if f is continuous everywhere except a finite number of points.*

2.1 Shortcomings

- Many functions are not integrable. For example the Dirchlet function (figure 6):

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{elsewhere} \end{cases}$$

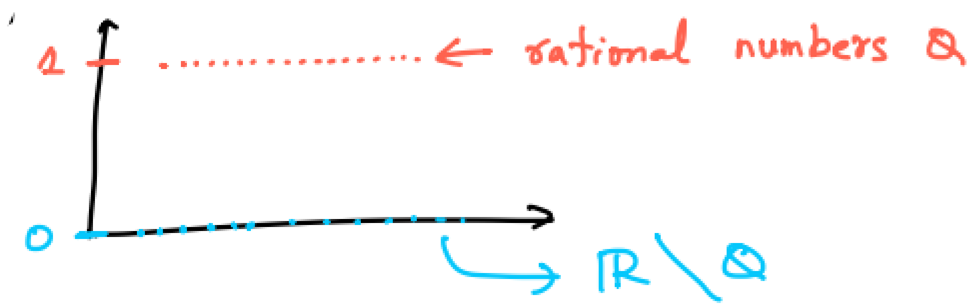

Figure 6: Dirchlet function is not integrable

as for any interval $I_k = [x_k, x_{k+1}]$ $M_k = 1$ and $m_k = 0$. Meaning $J_* = |b-a| \cdot 0 < J^* = |b-a| \cdot 1$

- One cannot prove theorems about exchanging "integral" with "lim": $\lim_{n\to\infty} \int f_n\,dt \stackrel{?}{=} \int \lim f_k\,dt$
- Hard to extend to "other space" (e.g. spaces with no notion of ordering, higher dimensional)

A more modern interpretation which solves many of these problems is **Lebesque Integration** which we will study later in the course

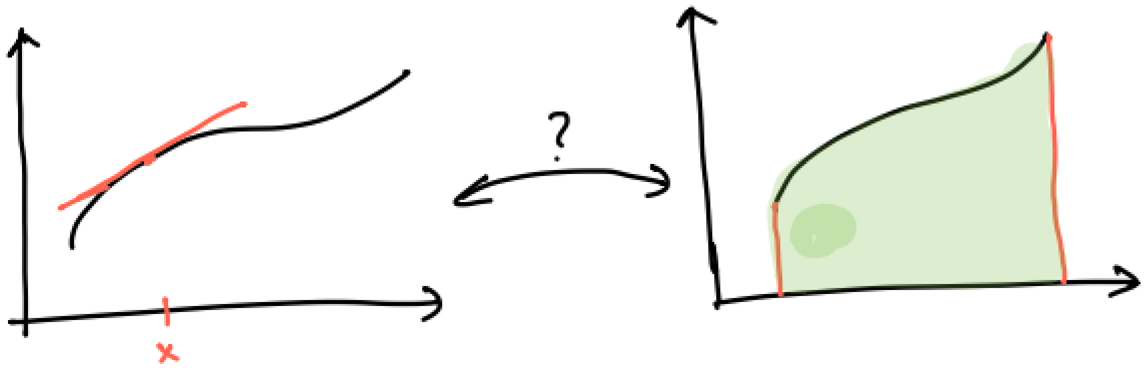

Figure 7: The relationship between the derivative (left) and integral (right) is not obvious

3 Fundamental Theorem of Calculus

Theorem 11 *$f : [a, b] \to \mathbb{R}$ (Riemann)-integrable and continuous at $\zeta \in [a, b]$. Let $c \in [a, b]$. Then the function,*

$$F(x) := \int_c^x f(t)\, dt$$

is differentiable at ζ and $F'(\zeta) = f(\zeta)$. If $f \in \mathscr{C}([a, b])$ (continuous), then $F \in \mathscr{C}^1([a, b])$ (continuous and once differentiable) and $F'(x) = f(x)$ for all $x \in [a, b]$

Theorem 12 *$F : [a, b] \to \mathbb{R}$ continuously differentiable, then*

$$\int_a^b F'(t)\, dt = F(b) - F(a)$$

Proof of Theorem 11: Need to prove that $F(x)$ is differentiable at ζ. Consider

$$A(h) := \frac{F(\zeta + h) - F(\zeta)}{h} = \frac{1}{h}\left(\int_c^{\zeta+h} f(t)\, dt - \int_c^{\zeta} f(t)\, dt\right) = \frac{1}{h}\int_\zeta^{\zeta+h} f(t)\, dt$$

We want to prove that this converges to $f(\zeta)$ as $h \to 0$, which can be expresses as wanting to show $A(h) - f(\zeta) \to 0$. Notice that $f(\zeta) = \frac{1}{h}\int_\zeta^{\zeta+h} f(\zeta)\, dt$ as $f(\zeta)$ is a constant. Thus,

$$A(h) - f(\zeta) = \frac{1}{h}\int_\zeta^{\zeta+h} f(t)\, dt - f(\zeta) = \frac{1}{h}\int_\zeta^{\zeta+h} f(t)\, dt - \frac{1}{h}\int_\zeta^{\zeta+h} f(\zeta)\, dt = \frac{1}{h}\int_\zeta^{\zeta+h} (f(t) - f(\zeta))\, dt$$

Intuitively, this should be small as h goes to zero since f is continuous at ζ.

Formally: Given $\epsilon > 0$ we can find $h > 0$ such that $f(t) - f(\zeta) < \epsilon \forall t \in [\zeta, \zeta + h]$. Then:

$$\frac{1}{h}\int_\zeta^{\zeta+h} (f(t) - f(\zeta))\, dt \leq \frac{1}{h}\int_\zeta^{\zeta+h} |f(t) - f(\zeta)|\, dt \leq \frac{1}{h}\int_\zeta^{\zeta+h} \epsilon\, dt = \frac{1}{h} \cdot \epsilon \int_\zeta^{\zeta+h} 1\, dt$$

$$= \frac{1}{h} \cdot \epsilon \cdot h = \epsilon$$
$$\implies A(h) - f(\zeta) \leq \epsilon \to 0$$

□

Proof of Theorem 12: We know that F' is continuous. Then by Theorem 11 the function

$$G(x) := \int_a^x F'(x)dt$$

is differentiable and

(i) $G(a) = 0$ (by def. of G)

(ii) $G'(x) = F'(x)$ on $[a, b]$ (by Theorem 11)

Consider $H(x) := F(x) - G(x)$ By (ii) we know that $H'(x) = F'(x) - G'(x) = 0 \forall x$. Hence, H is a constant function. We know that $H(a) = F(a) - G(a) = F(a)$ (as $G(a) = 0$ by (i)) Giving us:

(iii) $H(x) \equiv F(a)$ (constant)

Consider $x = b$.

$$F(a) \overset{\text{(iii)}}{=} H(b) \overset{\text{def}}{=} F(b) - G(b) \overset{\text{def}}{=} \int_a^b F'(t)\,dt$$
$$\implies F(a) = F(b) - \int_a^b F'(t)\,dt$$
$$\implies \int_a^b F'(t)\,dt = F(b) - F(a)$$

□