CSE 840: Computational Foundations of Artificial Intelligence October 11, 2023

Power Series

Instructor: Vishnu Boddeti *Scribe: Savvy Barnes*

1 Introduction

This lecture covers Power Series and Taylor Series.

2 Power Series

Definition 1 *A series of the form $p(x) : \sum_{n=0}^{\infty} a_n x^n$ is called a power series, where "power" is "power" of x and "series" is an infinite sum.*

Theorem 2 *For every power series $p(x) : \sum_{n=0}^{\infty} a_n x^n$, there exists a constant r, $0 \le r \le \infty$, called the **radius of convergence**, such that:*

- *The series converges (absolutely) for all x with $|x| < r$.*
- *If $|x| < r$, then the series converges uniformly.*

The first point above means that when $\sum_{n=0}^{\infty} a_n |x|^n$ converges, the sequence of partial sums $P_N(x) := \sum_{n=0}^{N} a_n |x|^n$ converges. However, it is unclear what happens when $|x| = r$, and when $|x| > r$, it is likely to diverge.

The radius of convergence only depends on $(a_n)_n$ and can be computed by various formulas, such as the two shown below, if the limit exists:

- $r = \frac{1}{L}$ where $L = \limsup_{n\to\infty} (|a_n|)^{\frac{1}{n}}$
- $r = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$

2.1 Example

Find the radius of convergence for: $p(x) : \sum_{n=0}^{\infty} n^c x^n$ for some constant c.
$r = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{n^c}{(n+1)^c} = \lim_{n\to\infty} (\frac{n^c}{n+1})^c = 1$
Now we need to check different values of x, to make sure it converges when $|x| < r$ or, in our case, $|x| < 1$. We can also see what happens for other instances of x, since our theorem tells us nothing about those. We will also need to consider different cases of c.

- Case $c = 1 \to \sum \frac{1}{n} x^n$ has radius of convergence of 1
  - For x = 1, the series diverges.
    $\sum \frac{1}{n} x^n = \sum \frac{1}{n} * 1^n = \sum_{n=0}^{\infty} \frac{1}{n} \to \infty$

- For x = -1, it converges.
- For x > 1, it diverges.

- Case $c = 0 \rightarrow \sum n^c x^n = \sum x^n$
  - For x = 1 and -1, the series diverges ($|x| = r$).

2.2 Exponential Series

This is a very easy Power Series, expressed as

$$exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

has $r = \infty$ since $\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = n + 1 \rightarrow \infty$ (using the same formula as before). Another famous Power Series is: $\sum_{n=0}^{\infty} n!x^n$, which has a $r = 0$ since $\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} \rightarrow 0$

3 From Power Series to Taylor Series

Observation 3 *Given power series $f(x_0 + h) = \sum_{n=0}^{\infty} a_n h^n$, we can take its derivative:*

$$f'(x_0 + h) = (a_0 + a_1 h + a_2 h^2 + ...)'$$
$$= a_1 + 2a_2 h + 3a_2 h^2 + ...$$
$$\sum_{n=1}^{\infty} n a_n h^{n-1}$$
$$f"(x_0 + h) \rightarrow f^{(K)}(x_0 + h) = \sum_{n=K}^{\infty} a_n (n * (n-1) * (n-2)...(n-k+1)) h^{(n-k)}$$

In all, we have

$$f^{(K)}(x_0) = a_K K! \rightarrow a_K = \frac{f^{(K)}(x_0)}{K!}$$

We began with a particular definition of a Power Series, and we did not know what the coefficients were. They were denoted by a_n, but not actually represented. By taking the derivatives at point x_0 we were able to find the coefficients. The next theorem below explains this.

Theorem 4 *Let $f(x_0 + h) = \sum_{n=0}^{\infty} a_n h^n$ with $r > 0$. Then for h with $|h| < r$, we have:*

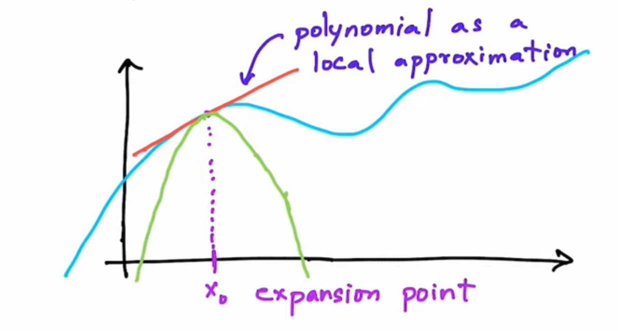
$$f(x_0 + h) = \sum_{n=0}^{\infty} \frac{f^{(n)(x_0)}}{n!} h^n$$

The intuition is that we begin with a Power Series that converges, which gives us a nice formulas that expresses the coefficients in terms of the derivatives of the function.

The reverse of this theorem (given a function and building a series that converges to it) is possible under some conditions - leading us to the Taylor Series.

4 Taylor Series

Taylor Series is used to approximate functions through their derivative. We are approximating a function at its expansion point with a polynomial, shown in the image below.



There are two ways we can look at this approximation:

1. **Linear Approximation (Line)**
   Consider a new point: $(x = x_0 + h)$
   $f(x_0 + h) = f(x_0) + f'(x_0)h + r(h)h$ with $r(h) \xrightarrow{h\to 0} 0$
2. **Quadratic Approximation (Parabola)**
   $f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f"(x_0)h^2 + r(h)h^2$ with $r(h) \xrightarrow{h\to 0} 0$

Theorem 5 *$I \subset \mathbb{R}$ open interval, $f : I \to \mathbb{R}$. $f \in \mathcal{C}^{n+1}([a,b])$, $x_0 \in I$. Define $T_n(x_0, h) := \sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!} * h^K$ where h^K is the Taylor Series up to degree n, and the numerator is differentiable. $R_n(x_0, h) := \int_{x_0}^{x_0+h} \frac{(x+h-t)^n}{n!} f^{(n+1)}(t)\,dt$ where $f^{(n+1)}$ is the remainder term. Then $f(x_0 + h) = T_n(x_0, h) + R_n(x_o, h)$.*

Proof: (Sketch)
The proof follows from the fundamental theorem of calculus, by induction on n. To do induction, we start with a base case, then prove an inductive step.
Base Case $n = 0$: we need to prove $f(x_0 + h) = f(x_0) + \int_{x_0}^{x_0+h} f'(t)\,dt$. This is similar to the fundamental theorem of calculus ($[\int_a^b F'(x)\,dx = F(b) - F(a)]$)
Inductive Step $n \to n + 1$:

- Consider: $F(x_0 + h) = \frac{(x_0+h-t)^{(n+1)}}{(n+1)!} f^{(n+1)}(t)$
- Take its derivative.
- Integrate and exploit fundamental theorem.

$\square$

The next theorem is the Taylor with Lagrange Remainder Theorem. It is a bit more useful, so we will show the whole proof.

Theorem 6 *$I \subset \mathbb{R}$, $f : I \to \mathbb{R}, f \in \mathcal{C}^{n+1}(I), x_0 \in I$. If $h \in \mathbb{R}$ such that $x_0 + h \in I$, then:*

$$f(x_0 + h) = \sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!} * h^K + R_n(h)$$

(where the first half of the sum is the n-th order Taylor polynomial and the second is the remainder term) and there is ξ with $\xi \in (x_0, x_0 + h)$ on $\xi \in (x_0 + h, x_0)$ such that $R_n(h) = \frac{f^{(n+1)}\xi}{(n+1)!} * h^{(n+1)}$.

Remark 7 *We often write $f(x_0+h) = \sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!} * h^K + O(h^{(n+1)})$ or with $x = x_0+h$ we write $f(x) = \sum_{K=0}^{n} \frac{f^{(K)}(x_0)}{K!} * (x - x_0)^K + O((x - x_0)^{(n+1)})$*

Proof: Define two quantities:

1. $F_{n,h}(t) := \sum_{K=0}^{n} \frac{f^{(K)}(t)}{K!}(h + x_0 - t)^K$
   Note: $F_{n,h}(x_0) := T_n(x_0, h), F_{n,h}(x_0 + h) = f(x_0 + h)$
2. $G_{n,h}(t) := (h + x_0 - t)^{(n+1)}, G'_{n,h}(t) = -(n + 1) * (h + x_0 - t)^n$

Now we apply the Generalized Mean Value Theorem

$$\frac{F_{n,h}(x_0 + h) - F_{n,h}(x_0)}{G_{n,h}(x_0 + h) - G_{n,h}(x_0)} = \frac{F'_{n,h}(\xi)}{G'_{n,h}(\xi)}$$

where $\xi \in (x_0, x_0 + h)$.
Putting it all together, we have:

$$f(x_0 + h) - T_n(x_0 + h) = (G_{n,h}(x_0 + h) - G_{n,h}(x_0)) * \frac{F'_{n,h}(\xi)}{G'_{n,h}(\xi)}$$

where $G_{n,h}(x_0 + h)$ is O and $G_{n,h}(x_0)$ is h^{n+1}.

$$= \frac{h^{n+1} * F'_{n,h}(\xi)}{(n + 1)(h + x_0 - \xi)^n}$$

We need to calculate $F'_{n,h}(t)$ and plug it back in:

$$F'_{n,h}(t) = \frac{d}{dt} \sum_{K=0}^{n} \frac{f^{(K)}(t)}{K!}(h + x_0 - t)^K$$

$$= \sum_{K=0}^{n} \frac{f^{(K+1)}(t)}{K!}(h + x_0 - t)^K - \sum_{K=1}^{n} \frac{f^{(K)}(t)}{(K - 1)!}(h + x_0 - t)^{K-1}$$

$$= \frac{f^{n+1}(t)}{n!} * (h + x_0 - t)^n$$

$$= \frac{h^{n+1} * \frac{f^{n+1}(t)}{n!} * \cancel{(h + x_0 - t)^n}}{(n + 1) * \cancel{(h + x_0 - t)^n}}$$

$$= \frac{h^{n+1} * f^{(n+1)}(\xi)}{(n+1)!}$$

$$f(x_0 + h) = T_n(x_0 + h) + \frac{h^{n+1} * f^{(n+1)}(\xi)}{(n+1)!}$$

$\square$

If we look back up at the remainder above, that's what we have here.

Theorem 8 *$f \in \mathcal{C}^\infty(I), x_0 \in I, h \in \mathbb{R}$ such that $x_0 + h \in I$. Define:*

$$T(x_0, h) := \lim_{n \to \infty} T_n(x_0, h) = \sum_{h=0}^{\infty} \frac{f^n(x_0)}{n!} * h^n$$

Then we have $f(x) = T(x)$ if $R_n(x_0, h)^n \xrightarrow{n \to \infty} 0$

For example, this is the case if there exists constants $\alpha, c > 0$ such that $|f^n(t)| \leq \alpha * c^n, \forall t \in I, \forall n \in, \mathbb{N}$ where $|f^n(t)|$ is a sufficient but not necessary condition (this is not the only way this term may go to 0, but if this condition holds, it goes to 0). The proof for this follows directly from the Lagrangian remainder theorem that we just did.

Now we have the full set of Taylor Theorems, so we can represent any function f as a sum of the decomposition of its Taylor polynomial and remainder. If the remainder converges to 0 as $n \to \infty$, then the function is equal to the Taylor polynomial. For the remainder to converge to 0, the n-th derivative has to be bounded as less than or equal to α times c to the power of n.

Not all functions have a Taylor Series that converges to the function.

4.1 Examples

- **Exponential Series**:
  $exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a Power Series with $r = \infty$. It will always coincide with its Taylor Series. Other examples include $\sin, \cos$, polynomials, Power Series. These are called **analytic functions**.

- $f(x) = \log(x+1)$ Taylor Series around 0, $r = 1$.
  For x outside of $(-1, 1)$, the Taylor series does not make sense.

- **Taylor Series Does NOT Coincide with Function**
  $f(x) = \begin{cases} exp(-\frac{1}{x^2}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$
  It has the weird property that $\forall n \in \mathbb{N}$: $f^n(0) = 0$. If we consider the Taylor Series of it about $x_0 = 0$, then all terms will be 0, i.e. $\forall n : T_n(0, h) = 0$ and $r = \infty$.

  $$f(x_0 + h) = T_n(x_0, h) + R_n(x_0, h)$$

  We are approximating around the point (small ball), not just at the point.

  $$T_n(x_0 = 0, h) = 0 \text{ but } f(0 + h) = exp(\frac{-1}{h^2})$$

On the left, we see the Taylor Series around $x_0 = 0$ is 0, but the right side shows that function values around $x_0 = 0$ are not 0. As an equation:

$$\forall (x_0 + h) \neq 0; T_n(x_0, h) \neq f(x_0 + h)$$