σ - Algebra, Measure

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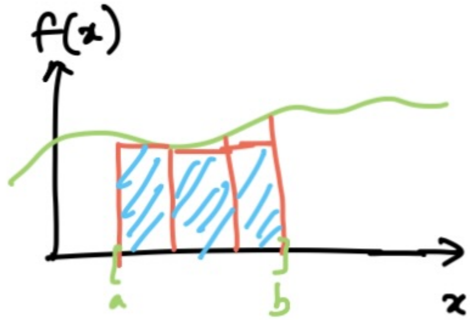
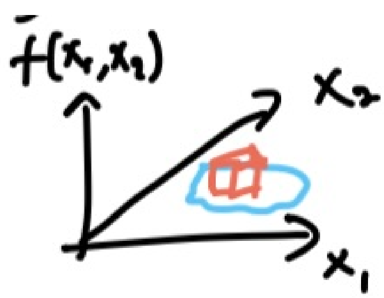
Reimann Integral



$f : \mathbb{R} \to \mathbb{R}$
$\int_a^b f dt \approx \sum_k vol(I_k).f(m_k)$
Here, $vol(I_k) = x_{k+1} - x_k$

Problems of Reimann Integral:

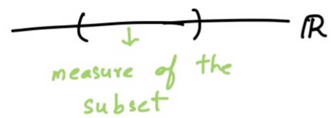


(i) Difficult to extend to higher dimensions.
(ii) Dependence on continuity.
(iii) Limit processes.

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

Our Goal is to get to Lebesgue Integrals

Lebesgue Integrals



Let X be a set, P(X) be the power set of X
Example : $X = \{a, b\}, P(X) = \{\phi, X, \{a\}, \{b\}\}$

Def: $\mathbb{A} \subseteq$ P(X) is called σ - Algebra:
(a) ϕ, X $\in \mathbb{A}$
(b) A $\in \mathbb{A}$ => A^c :=x/A $\mathbb{A} \in \mathbb{A}$
(c) $A_i \in \mathbb{A}$, i $\in \mathbb{N} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathbb{A}$

Def: A Measurable space consists of a set X and a σ - Algebra $\mathbb{A}$ over X.
Notation: (X,$\mathbb{A}$).
The sets A $\in \mathbb{A}$ are called $\mathbb{A}$-measurable sets.

Examples:
(1) $\mathbb{A} = \{\phi$,X$\}$ $\rightarrow$ smallest
(2) $\mathbb{A}$ = P(X) $\rightarrow$ largest

Let $\mathbb{A}_i$ be a σ - algebra on X, i $\in$ I(index set)
Then $\bigcap_{i \in I} \mathbb{A}_i$ is also a σ - algebra on X.

Def : For M $\subseteq$ P(x) , there is a smallest σ - algebra that contains M :
(a)σ (M) := $\bigcap$ A denotes the σ-algebra generated by M
(b)A $\supseteq$ M

Example:
Let X = {a,b,c,d}
Let M = {{a},{b}}
σ (M)= {ϕ, X ,{a},{b},{a,b},{b,c,d},{a,c,d},{c,d}}

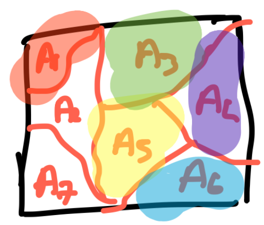
Def: Let (X, Υ) be a topological space or
(Let X be a metric space) or
(Let X be a subset of $\mathbb{R}^n$)

Remark: We need open sets for this.
B(X) is called the Borel σ - algebra on X
Note : This is the σ -algebra generated by the open sets
B(X) := $\sigma(\Upsilon)$

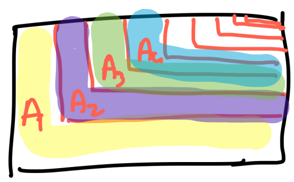
Measures

Def: Let (X,$\mathbb{A}$) be a measurable space. Consider a map $\mu : \mathbb{A} \rightarrow [0,\infty]$ is called a measure if it satisfies:

(a) $\mu(\phi) = 0$



(b) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ with $A_i \bigcap A_j \neq \phi$
Note : i $\neq$ j for all $A_i \in \mathbb{A}$



Def : A measurable space (X,$\mathbb{A}$) endowed with a measure μ is called a measure space (X,$\mathbb{A}$,μ)

Examples

- For the following examples, lets consider the set X and the $\sigma - Algebra\ A$ be the power set P(X).

$\Rightarrow$ X, $\mathcal{A} = P(X)$

a) Counting measure:
The counting measure is defined as follows:

$$\mu(A) := \begin{cases} \text{Number of elements in A}, & \text{A has finitely many elements} \\ \infty & \text{else} \end{cases}$$

$(A \in \mathcal{A})$

Calculation rules in [0,∞]:

- x + ∞:= ∞ $\forall$ x$\in$ [0,∞]
- x . ∞:= ∞ $\forall$ x$\in$ (0,∞]
- 0 . ∞:= 0 (!in most cases of measure theory!)

b) Dirac measure for p $\in$ X:

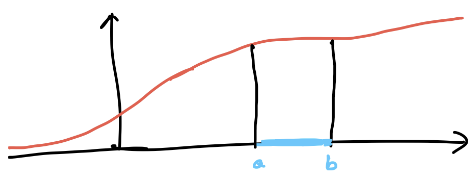
$$\delta_p := \begin{cases} 1, & p \in A \\ 0, & else \end{cases}$$

c) We want to define a measure on X=$\mathbb{R}^n$

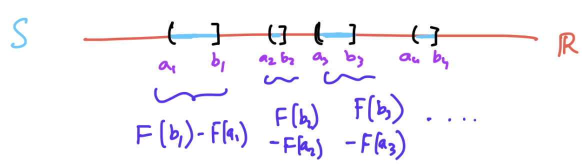
- $\mu([0,1]^n) = 1$
- $\mu(X + A) = \mu(A)$ $\forall$ x $\in$ $\mathbb{R}^n$

d) A more useful class of measures on $\mathbb{R}^n$.

X = $\mathbb{R}$,$\mathcal{A}$ Borel σ-algebra. Let F:$\mathbb{R} \to \mathbb{R}$ be a monotonically increasing, continuous function.



We define a measure μ_F on ($\mathbb{R}$,$\mathcal{A}$) by setting μ_F(S) = $inf\{\sum_{j=1}^{\infty} F(b_j) - F(a_j) | S \subset U_{j=1}^{\infty}(a_i, b_j]\}$



The general procedure to computer this measure is given by:

- Cover S by intervals.
- To each interval we assign 'elementary volume' which is given by F(b)-F(a).
- Take best covering which is given by the infimum of all the possible coverings of S.

$\to$ Need to prove: this is a measure!
A subset N $\in$ $\mathcal{A}$ is called a **null set** if $\mu(N) = 0$.

- We say a property holds almost everywhere if it holds $\forall$x $\in$ X except for x in a null set N.