CSE 840: Computational Foundations of Artificial Intelligence	Oct 30, 2023

Lebesgue Measure on $\mathbb{R}^n$, A set that is not Lebesgue measurable

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1 The Lebesgue Measure on

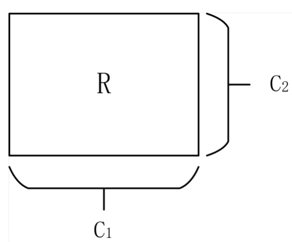
Want to construct a measure on $\mathbb{R}^n$. Want that rectangles of the form

$$[a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$$

have the "natural volume" given by

$$\prod_{i=1}^{n} (b_i - a_i)$$

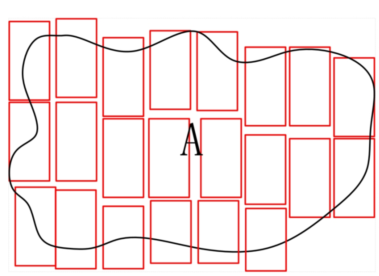
For a rectangle R with sides c_1 and c_2:



$$\text{vol}(R) := c_1 \cdot c_2$$

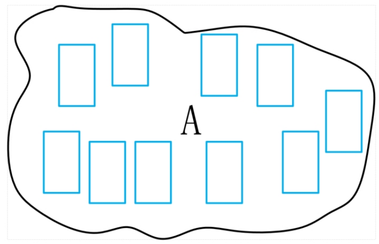
First approaches (Jordan, Riemann) attempted the following:

"Outer approximation":



$$A \subsetneq \bigcup_{i=1}^{n} \text{rectangle}_i$$

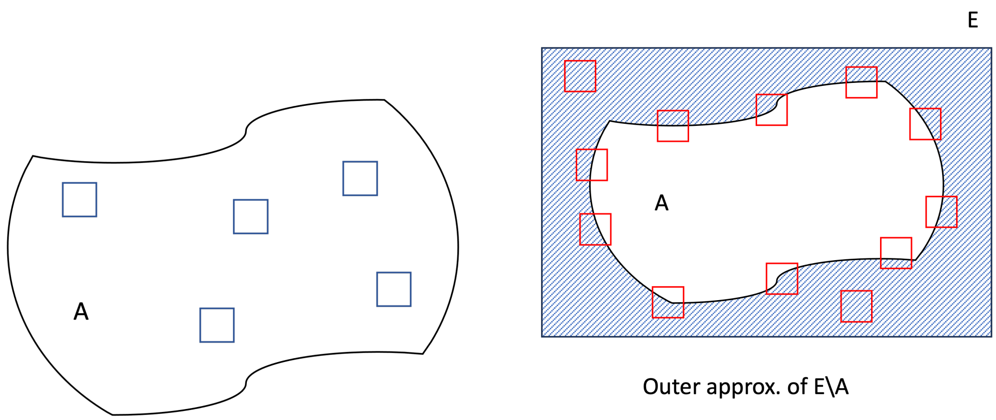
"Inner approximation":



$$\bigcup_{i=1}^{n} \text{rectangle}_i \subsetneq A$$

A would be called "measurable" if outer and inner approximation converges.

- Allow for countable coverings.
- Replace inner approximation by an outer approximation of the complement.



Outer approx. of E\A

Given a set E and its subset A, we have:

$$\mu(E) = \mu(E \setminus A) + \mu(A)$$
$$\implies \mu(A) = \mu(E) - \mu(E \setminus A)$$

- Need σ-algebra as underlying structure.

2 Outer Lebesgue Measure

Set the "natural volume" of rectangles:

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$$

$$|R| := \prod_{i=1}^{n} (b_i - a_i)$$

Definition 1 ***Definition of outer Lebesgue measure:***

Let $A \subseteq \mathbb{R}^n$ be arbitrary. We define

$$\lambda(A) := \inf \left\{ \sum_{i=1}^{\infty} |R_i| \mid A \subsetneq \bigcup_{i=1}^{\infty} R_i, \text{ where } R_i \text{ is a rectangle} \right\}$$

We cover A by a countable union of rectangles, then take infimum. Observe:

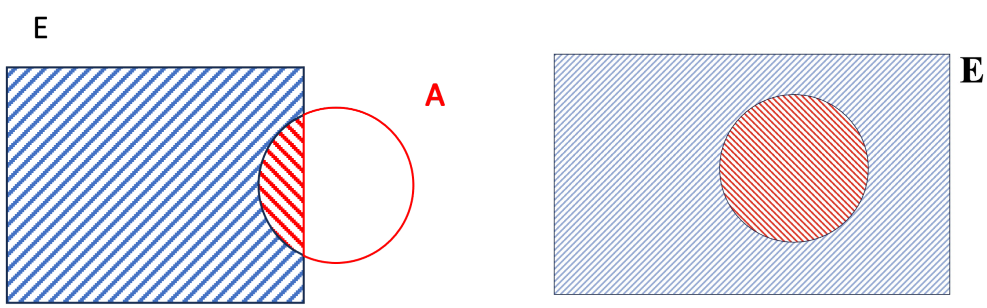
$$\lambda(A) \in [0, \infty) \cup \{\infty\}$$

We want to make $\lambda(A)$ into a measure.

Problem: If we use $P(\mathbb{R}^n)$ as σ-algebra, we run into contradictions. We need to restrict ourselves to a smaller σ-algebra.

Definition 2 *We say that a set $A \subseteq \mathbb{R}^n$ is measurable if, for every $E \subseteq \mathbb{R}^n$:*

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \setminus A)$$



Denote by $\mathcal{L}$ all measurable subsets of $\mathbb{R}^n$.

Theorem 3 *The set $\mathcal{L}$ forms a σ-algebra on $\mathbb{R}^n$.*

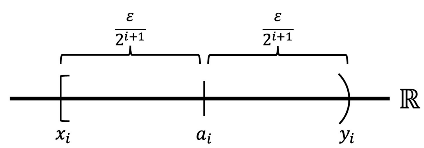
The outer measure λ is in fact a measure on $(\mathbb{R}^n, \mathcal{L})$. On rectangles, it coincides with the "natural volume".

Examples:

- $\lambda(\{x\}) = 0$
- $\lambda(\mathbb{R}) = \infty$
- If $A \subseteq \mathbb{R}$ is countable, then $\lambda(A) = 0$. In particular, $\mathbb{Q}$ is measurable and $\lambda(\mathbb{Q}) = 0$.

Proof: For $\varepsilon > 0$, define for all $a_i \in A$ the interval $[x_i, y_i)$ such that:

$$x_i = a_i - \frac{\varepsilon}{2^{i+1}} \quad \text{and} \quad y_i = a_i + \frac{\varepsilon}{2^{i+1}}$$



Then,

$$A \subseteq \bigcup_{i=1}^{\infty} [x_i, y_i]$$

We have:

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda([x_i, y_i])$$

which simplifies to:

$$\sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \varepsilon$$

Taking the inf. over all coverings, shows that:

$$\lambda(A) = 0$$

□

Comparison of $\mathcal{L}$ (σ-algebra of Lebesgue measurable sets) **with the Borel** σ **-algebra** $\mathcal{B}$:

(1) $\mathcal{B} \subseteq \mathcal{L}$:

- Open intervals are measurable, thus in $\mathcal{L}$.
- Any open set A in $\mathbb{R}^n$ can be written as a countable union of open intervals:

$$A \subseteq \bigcup_{i=1}^{\infty} I_i, \quad I_i \text{ is an open interval.}$$

(2) For every Lebesgue-measurable set L, there exists a set $B \in \mathcal{B}$ and $N \in \mathcal{L}$ with $\lambda(N) = 0$ such that $L = B \cup N$.

Summary: $\mathcal{L} \approx \mathcal{B}$ (up to sets of measure 0).

3 A non-measurable set

Measure problem: Search for a measure μ on $P(\mathbb{R})$ with the following properties:

(1)

$$\mu([a,b]) = b - a \quad \text{where} \quad b > a$$

(2)

$$\mu(x + A) = \mu(A) \quad \text{for all} \quad A \in P(\mathbb{R}), \quad x \in \mathbb{R}$$

$\Rightarrow \mu$ does not exist.

Claim 4 *Let μ be a measure on $P(\mathbb{R})$ with $\mu([0,1]) < \infty$ and (2). $\Rightarrow \mu = 0$.*

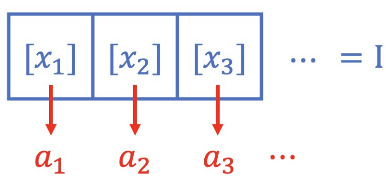
Proof:

(A) Definitions: Let $I = (0,1]$ with an equivalence relation on I

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$

i.e.

$$[x] := \{x + r \mid r \in \mathbb{Q}, x + r \in I\}$$



disjoint decomposition of I into boxes, possibly uncountable many of them!
$A \subseteq I$ with properties:

(i) For each $[x]$, there is an $a \in A$ with $a \in [x]$.
(ii) For all $a, b \in A$: $a, b \in [x] \Rightarrow a = b$.

$A = \{a_1, a_2, \dots\}$, we need the axiom of choice of set theory.
$A_n := r_n + A$, where $(r_n)_{n \in \mathbb{N}}$ enumeration of $\mathbb{Q} \cap (-1, 1]$.

(b) **Claim 5**

$$A_n \cap A_m = \emptyset \Leftarrow n \neq m$$

Proof:

$$x \in A_n \cap A_m \Rightarrow x = r_n + a_n, a_n \in A$$
$$x = r_m + a_m, a_m \in A$$
$$\Rightarrow r_n + a_n = r_m + a_m \Rightarrow a_n - a_m = r_m - r_n \in \mathbb{Q} \Rightarrow a_n \sim a_m$$
$$\Rightarrow a_n \in [a_m] \Rightarrow a_n = a_m \Rightarrow r_m = r_n \Rightarrow n = m$$

□

(c) **Claim 6**

$$(0,1] \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq (-1,2]$$

Proof: Exercise for you! □

Now assume:

μ is a measure on $P(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2).

By (2): $\mu(r_n + A) = \mu(A), \forall n \in \mathbb{N}$

By (C): $\mu((0,1]) \leq \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \mu((-1,2])$

We know:

$$\mu((0,1]) =: C < \infty$$

$$\mu((-1,2]) = \mu((-1,0] \cup (0,1] \cup (1,2]) = 3C$$

$$\text{(by using (2) and } \sigma\text{-addivity)}$$

$$\Rightarrow C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C$$

$$\Rightarrow C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3C$$

(i) $\mu(A) = 0$, $\sum_{n=1}^{\infty} \mu(A) = 0 \Rightarrow C = 0$

(ii) $\mu(A) > 0$, $\sum_{n=1}^{\infty} \mu(A) = \infty$, ~~$C \leq \infty \leq 3C$~~

$\Rightarrow \mu(A) = 0$

$$\mu(A) = 0 \Rightarrow C = 0 \quad (\text{ hence } \mu((0,1]) = 0)$$

$$\mu(\mathbb{R}) = \mu(\bigcup_{m\in\mathbb{Z}} (m, m+1]) = 0$$

$$\Rightarrow \mu = 0$$

□