1 Probability Measure

Definition 1

- Given space \( \Omega \) ("abstract space")
- Need a \( r \)-algebra \( A_r \) on \Omega \). ("measurable events")
  - \( A \in A_r \implies A^c \in A_r \)
  - \((A_i)_{i \in \mathbb{N}} \subset A_r \implies \bigcup_{i=1}^{\infty} A_i \in A_r \) ("countable unions")
  - \( \emptyset, \Omega \in A_r \)
  - countable intersections
- A measure \( \mu \) on \((\Omega, A_r)\) is a function \( \mu : A_r \to [0, \infty] \) that is countably additive: If \( (A_i)_{i \in \mathbb{N}} \) is a sequence of pairwise disjoint sets, then \( \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \)

A measure \( P \) on a measurable space \((\Omega, A_r)\) is called a probability measure if \( P(\Omega) = 1 \). The elements of \( A_r \) are called events. Then \((\Omega, A_r, P)\) is called a probability space.

Example (1):

Throw a die

\( \Omega = 1, 2, \ldots, 6, A_r = P(\Omega) \) (r-algebra generated by the "elementary events" \{1\}, \{2\}...\{6\}).

\( P \) can be defined uniquely by assigning \( P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = \frac{1}{6} \)

For example \( P(\{1, 5\}) = P(\{1\}) + P(\{5\}) = \frac{1}{3} \)

Throw two dice:

\( \Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\} = \{(1, 1), (1, 2), \ldots\} \) all of which are elementary events

\( A_r = P(\Omega) \)

\( P(\{(i, j)\}) = \frac{1}{36} \)

Example (2): Normal distribution

\( \Omega = \mathbb{R} \)

\( A_r = \text{Borel-r-algebra} \)

\( f_{\mu,r} : \mathbb{R} \to \mathbb{R} \)
Figure 1: $\mu$

Figure 2: $A$

\[ x \mapsto \frac{1}{\sqrt{2\pi}r} \exp\left(-\frac{(x-\mu)^2}{2r^2}\right) \]

\[ P : A_r \to [0, 1], P(A) := \int_A f_{\mu,r}(x)dx \]
2 Different Types of Probability Measures

Definition 1 Discrete measure:
\[ \Omega = \{x_1, x_2, \ldots \} \text{ finite and countable} \]
\[ A_r = P(\Omega) \]

We define a probability measure \( P : A_r \to [0, 1] \) by assigning probabilities to the "elementary events":
\[ P(\{x_i\}) =: P_i \]
with \( 0 \leq P_i \leq 1, \Sigma_i P_i = 1 \)

For \( A \in A_r \) we assign
\[ P(A) = \sum_{\{i \mid x_i \in A\}} P_i. \]

Examples: a coin toss, distribution on \( Q \)

Definition 2 Dirac measure:

For \( x \in \mathbb{R} \), we define the Dirac measure \( \delta_x \) on \((\mathbb{R}, B(\mathbb{R}))\) by setting
\[ \delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases} \]

sometimes this is called a point mass at a point \( x \). A discrete measure on \( \mathbb{R} \) can be written as a sum of Dirac measures. For example, throwing a die can be considered as
\[ \frac{1}{6}(\delta_1 + \delta_2 + \ldots + \delta_6) \]

Measures with a density

Consider \((\mathbb{R}^n, B(\mathbb{R}^n))\) and the Lebesgue measure \( \lambda \). Consider a function \( f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) that is measurable and satisfies
\[ \int f d\lambda = 1 \implies \int f(x) dx = 1. \]

Then we define a measure \( \gamma \) on \( \mathbb{R}^n \) by setting, for all \( A \in A_r \),
\[ \gamma(A) := \int_A f(x) dx \]

\( \gamma \) is the probability measure on \((\mathbb{R}^n, B(\mathbb{R}^n))\) with density \( f \).
Question: Can we describe every probability measure on $(\mathbb{R}^n, B(\mathbb{R}^n))$ in terms of density?

Answer: no!

Counterexample: $\delta_0$ Dirac measure.

On the same measure space $(\mathbb{R}^n, B(\mathbb{R}^n))$, if we have two measures $\lambda, \gamma$.

Question: $\gamma(A) = \int_A \emptyset d\lambda$

Does $\emptyset$ exist?

Answer: No!

**Definition 1.** A probability measure on $\gamma$ on $(\mathbb{R}^n, B(\mathbb{R}^n))$ is called absolutely continuous with respect to another measure $\mu$ on $(\mathbb{R}^n, B(\mathbb{R}^n))$ if every $\mu$-null set is also a $\gamma$-null set

$$\forall B \in B(\mathbb{R}^n) : \mu(B) = 0 \implies \gamma(B) = 0.$$

**Notation:** $\gamma \ll \mu$

$$\mu(A) = 0 \implies \int_A f d\mu = \gamma(A) = 0$$

**Example:** $N(0, 1) \ll \lambda$

$$\gamma_A = \int_A f d\lambda$$

**Example:** $\delta_0 \not\ll \lambda$ because

$$\lambda(0) = 0 \text{ but } \delta_0(0) = 1$$
Theorem 2. (Radon-Nikodym): Consider two probability measures \(\gamma, \mu\) on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\). Then the following two statements are equivalent:

\[
\begin{align*}
(1) & \quad \gamma \text{ has a density w.r.t } \mu. \\
(2) & \quad \gamma \text{ is absolutely continuous w.r.t } \mu.
\end{align*}
\]

If \(\gamma \ll \mu\), then \(\exists \phi\) such that \(\delta(A) = \int_A \phi \, d\mu\), \(\emptyset\) exists and is unique.

Proof idea:

(1) \(\implies\) (2) easy

(2) \(\implies\) (1) We need to construct a density!

Consider the set \(G\) of all functions \(g\) with the following properties:

\[
\begin{align*}
\bigoplus & \quad \begin{cases} \\
\quad g \text{ is measurable, } g \geq 0 \\
\quad g \ast \mu \leq \gamma, \text{ that is } \forall A \in \mathcal{B}(\mathbb{R}^n): \int_A g \, d\mu \leq \gamma(A).
\end{cases}
\end{align*}
\]

- Observe: \(g = 0\) satisfies \(\bigoplus\), so \(G\) is not empty.
- If \(g,h\) both satisfy \(\bigoplus\), then \(\sup(g,h)\) satisfies \(\bigoplus\).
- Define \(\xi := \sup_{g \in G} \int g \, d\xi\) and construct a sequence \((g_n)_{n \in \mathbb{N}}\) such that \(\lim \int g_n \, d\mu = \xi\).
- Define "density" \(f := sup g_n\).
- Now prove: \(f\) is the density that we are looking for. \(\square\)