1 Lebesgue Decomposition

**Definition 1** Consider a measure space \((X, A, \lambda)\) and another measure \(\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]\).

(a) \(\mu\) is called absolutely continuous if \(\lambda(A) = 0 \Rightarrow \mu(A) = 0\) for all \(A \in \mathcal{B}(\mathbb{R})\).

(b) \(\mu\) is called singular with respect to \(\lambda\) if there is \(N \in \mathcal{B}(\mathbb{R})\) with \(\lambda(N) = 0\) and \(\mu(N^c) = 0\).

**Theorem 2** Consider \(\mu, \gamma\) prob. measures on \((\Omega, \mathcal{A})\). Then there exists a unique decomposition \(\gamma = \gamma_{ac} + \gamma_s\) such that \(\gamma_{ac} \ll \mu\) and \(\gamma_s \perp \mu\).

Example: \(\gamma = \frac{1}{2}(N(0, 1), \delta_0)\). \(\gamma = \gamma_{ac} + \gamma_s\), where \(\gamma_{ac} = \frac{1}{2}N(0, 1), \gamma_s = \frac{1}{2}\delta_0\).

Cantor distribution: non-trivial distribution that is singular with respect to \(\lambda\). Construct the Cantor set:

- Start with \(C_0 := [0, 1]\)
- Start with \(C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\)

\[\vdots\]

The Cantor set is limited in this process. Now construct a prob. distribution:

Consider the CDFs of the sets \(C_0, C_1, C_2 \cdots\)
2 Cumulative Distribution Function

Let $P$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Define the function $F : \mathbb{R} \to \mathbb{R}, x \to P((-\infty, x])$. W says that $F$ is a cumulative distribution function (cdf), that satisfies the following properties:

(i) $F$ is monotonically increasing, $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

(ii) $F$ is continuous from the right: $(x_n)_{n \in \mathbb{N}}$ sequence with $x_n \leq x_{n+1}$ and $x_n \to x$ then also $F(x_n) \to F(x)$

Let $F : \mathbb{R} \to \mathbb{R}$ be a function with properties (i) and (ii). Then there exists a unique probability measure $P$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $P((-\infty, x]) := F(x)$
3 Random Variable

Definition 3 Let \((\Omega, A, P)\) be a probability space, \((\tilde{\Omega}, \tilde{A})\) be another measurable space. A mapping \(X : \Omega \rightarrow \tilde{\Omega}\) is called a random variable if \(X\) is measurable, i.e., \(\forall \tilde{A} \in \tilde{A}: X^{-1}(\tilde{A}) := \{w \in \Omega | X(w) \in \tilde{A}\} \in A\).

Definition 4 A random variable \(X : \Omega \rightarrow \tilde{\Omega}\) induces a measure on the target space: For \(\tilde{A} \in \tilde{A}\) we define \(P_X(\tilde{A}) := P(X^{-1}(\tilde{A}))\). This is a probability measure on \((\tilde{\Omega}, \tilde{A})\), and it is called the distribution of \(X\).

Definition 5 \(X : (\Omega, A, P) \rightarrow (\tilde{\Omega}, \tilde{A})\). Then the family \(\sigma(X) := \{X^{-1}(\tilde{A}) | \tilde{A} \in \tilde{A}\}\) is a \(\sigma\)-algebra induced by \(X\). (It is the smallest \(\sigma\)-algebra on \(\Omega\) that makes \(X\) measurable)

4 Conditional Probability

Notation: \(P(A \cap B) = P(\text{"A and B"})\); \(P(A \cup B) = P(\text{"A or B"})\)

Definition 6 Let \((\Omega, A, P)\) be a probability space. \(A, B \in A, P(B) > 0\). Then \(P(A|B) := \frac{P(A \cap B)}{P(B)}\) is called the conditional probability of \(A\) given \(B\).

Theorem 7 The mapping \(P_B : A \rightarrow [0, 1], A \rightarrow P(A|B)\) is a probability measure on \((\Omega, A)\), it is called the conditional distribution of \(P\) with respect to \(B\).

Examples:

1. two dice: \(P(\text{"sum is q"}|\text{"first die was 3"})\)
2. \(\Omega = \text{all persons on earth}, A = P(\Omega), P = \text{"uniform"}\).