Linear Mapping

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1 Linear Mapping

Definition 1 *Let U, V be vector spaces over the same field $\mathbb{F}$. A mapping $T : U \rightarrow V$ is called a linear map if $\forall \mathbf{u}_1, \mathbf{u}_2 \in U, \lambda \in \mathbb{F}$.*

$$t(\mathbf{u}_1 + \mathbf{u}_2) = t(\mathbf{u}_1) + t(\mathbf{u}_2)$$
$$t(\lambda \mathbf{u}_1) = \lambda t(\mathbf{u}_1)$$

The set of all linear mappings from $U \rightarrow V$ is denoted $\mathcal{L}(U, V)$.
If $U = V$, then we denote $\mathcal{L}(U)$.

Definition 2 *$T \in \mathcal{L}(U, V)$. Then kernel of T (null-space of T) is defined as*

$$ker(T) := null(T) := \{\mathbf{u} \in U | T\mathbf{u} = \mathbf{0}\}$$

Proposition 3

- *$ker(T)$ is a subspace of U*
- *T injective iff $ker(T) = \{\mathbf{0}\}$*

Definition 4 *The range of T (image of T) is defined as,*

$$range(T) := image(T) := \{T\mathbf{u} | \mathbf{u} \in U\}$$

Proposition 5

- *The range is always a subspace of V*
- *T is a subjective iff $range(T) = V$*

Definition 6 *Let V' be any subset of V, i.e. $V' \subset V$. The pre-image of V' is defined as*

$$T^{-1}(V') = \{\mathbf{u} \in U | T\mathbf{u} \in V'\}$$

Proposition 7 *If $V' \subset V$ is a subspace of V, then $T^{-1}(V')$ is a subspace of V*

Theorem 8 *Let V be finite-dim, W is any vector space, $T \in \mathcal{L}(V, W)$. Let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be a basis of $ker(T) \subset V$. Let $(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be a basis of $range(T) \subset W$. Then $\mathbf{u}_1, \ldots, \mathbf{u}_n, T^{-1}(\mathbf{w}_1), \ldots, T^{-1}(\mathbf{w}_m) \subset V$ form a basis of V. In particular, $dim(V) = dim(ker(T)) + dim(range(T))$.*

Proof: *Denote $T^{-1}(\mathbf{w}_1) = \mathbf{z}_1, \ldots, T^{-1}(\mathbf{w}_m) = \mathbf{z}_m$*

Step 1: $V \subset span\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{z}_1, \ldots, \mathbf{z}_m\}$

Let $\mathbf{v} \in V$ consider $T\mathbf{v} \in range(T)$.

$$\begin{aligned}
&\implies \exists \lambda_1, \ldots, \lambda_m \in \mathbb{F}, s.t. \\
&\quad T\mathbf{v} = \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \ldots + \lambda_m \mathbf{w}_m \\
&\quad = \lambda_1 T(\mathbf{z}_1) + \lambda_1 T(\mathbf{z}_2) \ldots + \lambda_1 T(\mathbf{z}_3) \\
&\quad = T(\lambda_1 \mathbf{z}_1 + \lambda_1 \mathbf{z}_2 \ldots + \lambda_1 \mathbf{z}_3)
\end{aligned}$$

$$\begin{aligned}
&\implies T\mathbf{v} - T(\lambda_1 \mathbf{z}_1 + \lambda_1 \mathbf{z}_2 \ldots + \lambda_1 \mathbf{z}_3) = 0 \\
&\implies T(\underbrace{\mathbf{v} - (\lambda_1 \mathbf{z}_1 + \lambda_1 \mathbf{z}_2 \ldots + \lambda_1 \mathbf{z}_3)}_{\in ker(T)}) = 0
\end{aligned}$$

$$\begin{aligned}
&\quad \textit{Reminder: } \mathbf{u}_1, \ldots, \mathbf{u}_n \textit{ are basis of } ker(T) \\
&\implies \exists \mu_1, \ldots, \mu_n \in \mathbb{F}, s.t. \\
&\quad \mathbf{v} - (\lambda_1 \mathbf{z}_1 + \lambda_1 \mathbf{z}_2 \ldots + \lambda_1 \mathbf{z}_3) = \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{u}_2 + \ldots + \mu_n \mathbf{u}_n \\
&\implies \mathbf{v} = \lambda_1 \mathbf{z}_1 + \lambda_1 \mathbf{z}_2 \ldots + \lambda_1 \mathbf{z}_3 + \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{u}_2 + \ldots + \mu_n \mathbf{u}_n
\end{aligned}$$

Step 2: $\mathbf{u}_1 \ldots \mathbf{u}_n, \mathbf{z}_1 \ldots \mathbf{z}_m$ are linearly independent.

Assume that

$$\mu_1 \mathbf{u}_1 + \ldots + \mu_n \mathbf{u}_n + \lambda_1 \mathbf{z}_1 + \ldots + \lambda_m \mathbf{z}_m = \mathbf{0} \tag{1}$$

Now consider:

$$\begin{aligned}
&\lambda_1 \mathbf{w}_1 + \ldots + \lambda_m \mathbf{w}_m \\
=&\lambda_1 T(\mathbf{z}_1) + \cdots + \lambda_m T(\mathbf{z}_m) \\
=&\lambda_1 T(\mathbf{z}_1) + \cdots + \lambda_m T(\mathbf{z}_m) + \underbrace{\mu_1 T(\mathbf{u}_1) + \ldots + \mu_n T(\mathbf{u}_n)}_{0} \\
=&T(\underbrace{\lambda_1 \mathbf{z}_1 + \lambda_1 \mathbf{z}_2 \ldots + \lambda_1 \mathbf{z}_3 + \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{u}_2 + \ldots + \mu_n \mathbf{u}_n}_{=0 \textit{ by (1)}}) \\
=&\mathbf{0}
\end{aligned}$$

$\implies \lambda_1 \mathbf{w}_1 + \ldots + \lambda_m \mathbf{w}_m = 0.$

$\implies \lambda_1 = \lambda_2 = \ldots = \lambda_m = 0,$ *since $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are basis.*

$\implies \mu_1 \mathbf{u}_1 + \ldots + \mu_n \mathbf{u}_n = 0$

$\implies \mu_1 = \mu_2 = \ldots = \mu_n = 0,$ *since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are basis.* $\square$

Proposition 9 *$T \in \mathcal{L}(V, V), V$ is finite-dim. Then the following statements are equivalent.*

1. *T is injective.*
2. *T is surjective.*
3. *T is bijective.*

Proof: *Direct consequence of theorem.* $\square$

Does not hold in ∞-dim spaces.

Additional material 10 *T is said to be injective or a monomorphism if any of the following equivalent conditions are true:*

- *T is one-to-one as a map of sets.*
- $kerT = 0_V$
- $dim(kerT) = 0$
- *T is left-invertible, which is to say there exists a linear map $S : W \to V$ such that ST is the identity map on V.*

For further readings, please refer to Wikipedia:
https://en.wikipedia.org/wiki/Linear_map.

2 Matrices and Linear Mapping

Notation:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = (a_{ij})_{i=1,\dots,m,j=1,\dots,n}$$

Proposition 11 *Consider $T \in \mathcal{L}(v, w), v, w$ finite-dim. Let $\mathbf{v}_1 \dots \mathbf{v}_n$ be a basis of V, $\mathbf{w}_1 \dots \mathbf{w}_m$ be a basis of w.*

- $\mathbf{v} = \lambda_1 \mathbf{v}_1 \dots + \lambda_n \mathbf{v}_n$
  $T(\mathbf{v}) = T(\lambda_1 \mathbf{v}_1 \dots + \lambda_n \mathbf{v}_n)$
  $= \lambda_1 T(\mathbf{v}_1) \dots + \lambda_n T(\mathbf{v}_n)$
- *Each image vector $T(\mathbf{v}_j)$ can be expressed in basis $\mathbf{w}_1 \dots \mathbf{w}_m$. There exists co-efficient $a_{1j} \dots a_{mj}$, s.t.*
  $$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + \dots + a_{mj}\mathbf{w}_m.$$
- *We can stack these coefficients in a matrix that* $\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$

Notation: Let $T : V \to W$ be linear, let $\mathcal{B}$ a basis of V, $\mathcal{C}$ basis of W, We denote by $M(T, \mathcal{B}, \mathcal{C})$ the matrix corresponding to T w.r.t. bases $\mathcal{B}$ and $\mathcal{C}$.

Proposition 12 *Convenient properties of matrices: Let V, W be vector spaces, and consider the bases fixed. Let $S, T \in \mathcal{L}(V, W)$.*

- $M(S + T) = M(S) + M(T)$
- $M(\lambda S) = \lambda M(S)$
- *For* $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_n \mathbf{v}_n$ *we have that* $T(V) = M(T0) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ *where* $(\mathbf{v}_1 \ldots \mathbf{v}_n)$ *is basis of* V
- $T : U \to V, S : V \to W$ *linear, then* $M(S \circ T) = M(S) \cdot M(T)$

Additional material 13 *Matrix Row Operation*

Matrix row operations are operations that can be applied to the rows of a matrix to transform it. There are three primary types of matrix row operations:

1. ***Scalar Multiplication:*** *Multiply a row by a nonzero scalar.*
2. ***Row Addition:*** *Add a multiple of one row to another row.*
3. ***Row Interchange:*** *Swap the positions of two rows.*

Example

Consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -1 \\ 5 & 2 & 0 \end{bmatrix}$$

We will perform the following row operations on matrix A:

1. *Multiply the first row by 2.*
2. *Subtract 5 times the first row from the third row.*
3. *Swap the second and third rows.*

After applying these operations, we obtain the transformed matrix B:

$$B = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 2 & 0 \\ 0 & 14 & -31 \end{bmatrix}$$

Matrix row operations are commonly used in Gaussian elimination and matrix row reduction to solve linear systems and manipulate matrices in various applications.

For further readings, please refer to Wikipedia:
https://en.wikipedia.org/wiki/Matrix_(mathematics)

3 Invertible maps and Matrices

Definition 14 *$T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that*

$$S \circ T = Id_V \text{ and } T \circ S = Id_W.$$

The map S is called the inverse of T, denoted by T^{-1}.

Remark 15 *Inverse maps exist and are unique.*

Proposition 16 *A linear map is invertible iff it is injective an subjective i.e. bijective.*

Proof: *"⇒":*

invertible $\implies$ injective:

Suppose $T(\mathbf{u}) = T(\mathbf{v})$. Then $\mathbf{u} = T^{-1}(T(\mathbf{u})) = T^{-1}(T(\mathbf{v})) = \mathbf{v} \implies \mathbf{u} = \mathbf{v} \implies$ injective.

injective $\implies$ invertible:

$\mathbf{w} \in W$. Then $\mathbf{w} = T(T^{-1}(\mathbf{w})) \implies w \in$ range of $T \implies$ subjective.

"⇐":

injective & subjective $\implies$ invertible:

Let $\mathbf{w} \in W$. There exists unique $\mathbf{v} \in V$. s.t. $T(\mathbf{u}) = \mathbf{w}$

Define the mapping: $S(\mathbf{w}) = \mathbf{v}$. Clearly have $T \circ S = Id$.

Let $\mathbf{v} \in V$, Then $T((S \circ T)\mathbf{v}) = (T \circ S)(T\mathbf{v}) = Id \circ T\mathbf{v} = T\mathbf{v}$

$\implies (S \circ T)\mathbf{v} = \mathbf{v} \implies S \circ T = Id \implies S$ is inverse of T

Linear mapping:

Let $\mathbf{w}_1, \mathbf{w}_2 \in W, \alpha \in \mathbb{F} : S(\mathbf{w}_1 + \mathbf{w}_2) = S(\mathbf{w}_1) + S(\mathbf{w}_2), S(\alpha \mathbf{w}_1) = \alpha S(\mathbf{w}_1)$

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$. s.t. $T(\mathbf{v}_i) = \mathbf{w}_i$. Then $S(\mathbf{w}_i) = \mathbf{v}_i$

$$\begin{aligned} S(\mathbf{w}_1 + \mathbf{w}_2) &= S(T(\mathbf{v}_1) + T(\mathbf{v}_2)) \\ &= S(T(\mathbf{v}_1 + \mathbf{v}_2)) \\ &= \mathbf{v}_1 + \mathbf{v}_2 \\ &= S(\mathbf{w}_1) + S(\mathbf{w}_2) \end{aligned} \qquad \begin{aligned} S(\alpha \mathbf{w}_1) &= S(\alpha T(\mathbf{v}_1)) \\ &= S(T(\alpha \mathbf{v}_1)) \\ &= \alpha \mathbf{v}_1 \\ &= \alpha S(\mathbf{w}_1) \end{aligned}$$

$\implies$ *S is a linear transform.* $\square$

Additional material 17 ***Matrix Inversion with Gaussian Elimination***

Consider a 2×2 *matrix* A*:*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

To find its inverse, A^{-1}*, we can use Gaussian elimination. Here are the steps:*

1. *Form the augmented matrix* $[A \,|\, I]$*:*

$$\begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 1 & 3 & | & 0 & 1 \end{bmatrix}$$

2. *Apply row operations to transform the left side into the identity matrix:*

$$\begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & | & -\frac{1}{2} & 1 \end{bmatrix}$$

3. *Divide the first row by* 1 *and the second row by* $\frac{5}{2}$*:*

$$\begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & 1 & | & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

4. *Subtract* $\frac{1}{2}$ *times the second row from the first row:*

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & | & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

After performing these row operations, the left side of the augmented matrix is the identity matrix and the right side is the inverse of matrix A*:*

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

So, the inverse of matrix A *is the matrix displayed above.*

For further readings, please refer to Wikipedia:
`https://en.wikipedia.org/wiki/Invertible_matrix`

4 Inverse Matrix

Definition 18 *A square matrix* $A \in \mathbb{F}^{n \times n}$ *is invertible if there exists a square matrix* $B \in \mathbb{F}^{n \times n}$ *such that:* $A \cdot B = B \cdot A = Id = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$
The matrix B *is called the inverse matrix and is denoted by* A^{-1}*.*

Proposition 19 *The inverse matrix represents the inverse of the corresponding linear map, that is* $T : V \to V$

$$M(T^{-1}) = (M(T))^{-1}$$

In particular, a matrix is invertible iff the corresponding map is invertible.

Remark 20

- *The inverse matrix does not always exist.*
- $(A^{-1})^{-1} = A$,$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- A^T *invertible* $\Leftrightarrow$ A *invertible*
  $(A^T)^{-1} = (A^{-1})^T$
- $A \in \mathbb{F}^{n\times n}$ *invertible* $\Leftrightarrow$ *rank*$(A) = n$

- *The set of all invertible matrices is called a general linear group:*
  $GL(n, F)$= $\{A \in \mathbb{F}^{n\times n}$ $|A$ *invertible*$\}$