CSE 840: Computational Foundations of Artificial Intelligence September 06, 2023

Transpose, Change of Basis, Rank of a Matrix, Determinant

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1.1 Transpose

Definition 1 *Given a matrix $A = (a_{ij})_{ij} \in F^{mxn}$, the transpose matrix is given by $(A^T)_{kj} = A_{jk}$*

$$A = \begin{pmatrix} 1 & -9 & 3 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$$

If $F = \mathbb{C}$, then the conjugate transpose matrix is defined as, $(A^*)_{ij} = \bar{A}_{ij}$
$x = a + ib$, $\bar{x} = a - ib$
Useful in the context of the adjoint of an operator.

1.2 Change of Basis

Theorem 2 *Consider the identity mapping $I : V \to V$, $x \longmapsto x$. Assume we fix a basis for v (both in source and target space), then the corresponding matrix defined as follows:* $M(I, \beta, \beta) = A = \begin{pmatrix} 1 & 0 & 0... \\ 0 & 1 & 0... \\ 0... & 0... & 1... \end{pmatrix}$

Now consider $A = a_{1,2}, , ..., a_n$ and $\beta = b_1, b_2, ..., b_n$ both bases of v. How does the matrix of the identity mapping $I : (v, A) \longmapsto (V, \beta)$ look like? Since β is a basis, we can express the vectors in A as a linear combination:
$a_1 = t_{11}b_1 + t_{21}b_2 + ... + t_{n1}b_n$
$a_2 = ...$
Now we form the corresponding matrix, $T = \begin{pmatrix} t_{11} & ... & t_{1n} \\ ... & ... & 0... \\ t_{n1} & 0... & t_{nn} \end{pmatrix}$ *This matrix*

represents the identity. In the basis A, the first basis vector a_1 *has the representation* $T = \begin{pmatrix}1\\0\\0\\...\end{pmatrix}$

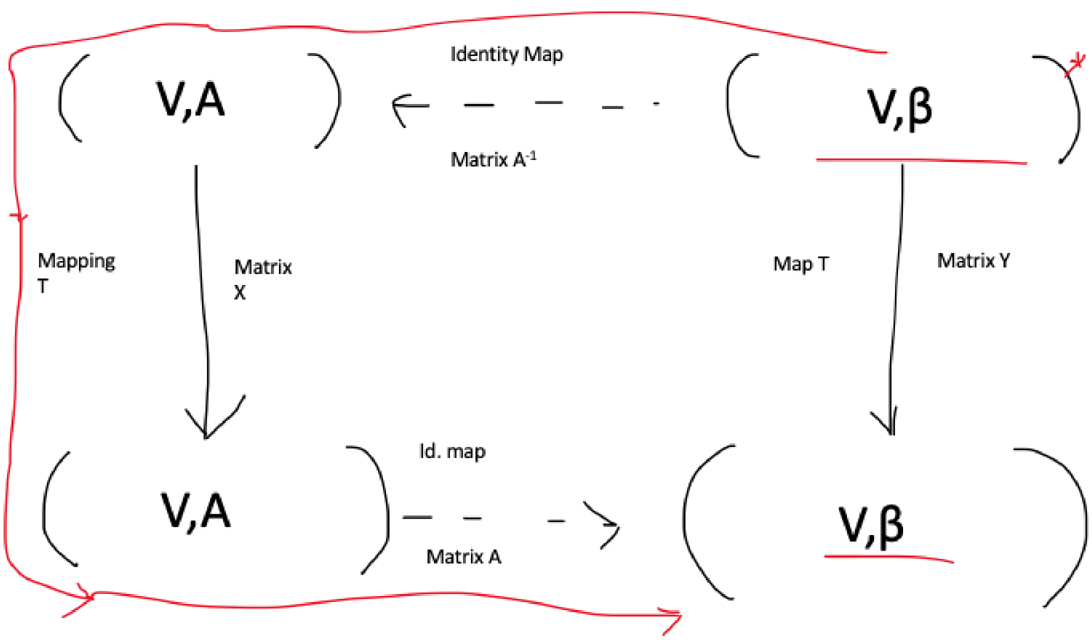
$a_1 = 1 * 1_1 + 0 * a_2 + ... + 0 * a_n$

$T\begin{pmatrix}1\\0\\0\\...\end{pmatrix} = \begin{pmatrix}t_{1!}\\t_{21}\\t_{n1}\\...\end{pmatrix} Thisvectorgivesus Ta, expressedinthebasis \beta.$

$t_{11} + g_{21}b_2 + ... + t_{n1}b_n = a_1$
$Ta_1 = [a_1]_\beta$

Proposition 3 *Let A, β bet two bases of V. Then the matrices M(I,A,β)andM(I,β, A) and invertible and each is the inverse of the others.*
$T_{A'} \longrightarrow_B = T^{-1}_B \longmapsto_A T^{-1} exists$

Proposition 4 *Let A, β be two bases of v. Consdier the transformation matrix A = M(I,A,β) and* A^{-1} *= M(I,β, A). Let T: v* $\longmapsto$ *v be a linear map and X: M(T,A,A). Then* $y := A * x * A^{-1}$ *represents T in the basis β, that is y = M(T β, β).*



1.3 Rank of a Matrix

Definition 5 $A \in F^{mxn}$. *The column Rank of A is dim(span column vectors of A)). The row rank is dim(span(row vectors of A).*

Proposition 6 *For a matrix, the rwo and column rank are always the same. We now call it the rank of the matrix.*

Proposition 7 $T \in L(V, W)$. *Then rank (M(T)) = dim (range(T)). Note, we did not specify any bases and the result holds independent of choice of basis.*

1.4 Determinant of a Matrix

Definition 8 *Consider a linear mapping d:* $F^{nxn} \longmapsto F$. *Then d is called a determinant if:*
(p1) d is a multilinear i.e. linear in each column of the matrix: Let A be a matrix of column $a_1, a_1, ..., a_n$. *Consider column* $a_i, assume a_i = a_1' + a_i''$ *for some* $a_i', a_i'' \in F^{nxn}$. *Then it holds that*
$det((a_1, ..., a_i, ..., a_n) = det((a_1, ..., a_i', ..., a_n) + det((a_1, ..., a_i'', ..., a_n) - det((a_1, ..., \lambda, ..., a_n) = \lambda((a_1, ..., a_i, ..., a_n)$
(p2) d is alternating: if A has two identical columns, then det(a) = 0
(p3) d is normed: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

Theorem 9 *The mapping d exists and is unique.*
Based on (p1),(p2),(p3), we can now prove many important propoerties of the determinant:
- The determinant of a linear mapping does not depend on the basis. $det(c*A) = c^n det(A)'A \leftarrow F^{nxn}$
*- det(A*B) - det(A) * det(B)*
- $det(A^T) = det(A)$
$-det(A^{-1}) = 1/det(A)$ *(if A is invertible)*
- A intertible $\leftrightarrow$ *det(A) != 0*
- det(A+B) != det(A) + det(B)
- If A is an upper triangular, $\begin{pmatrix} \lambda_1 & * \\ * & \lambda_n \end{pmatrix}$ *then* $det(A) = \lambda_1 * \lambda_2 * ...\lambda_n$

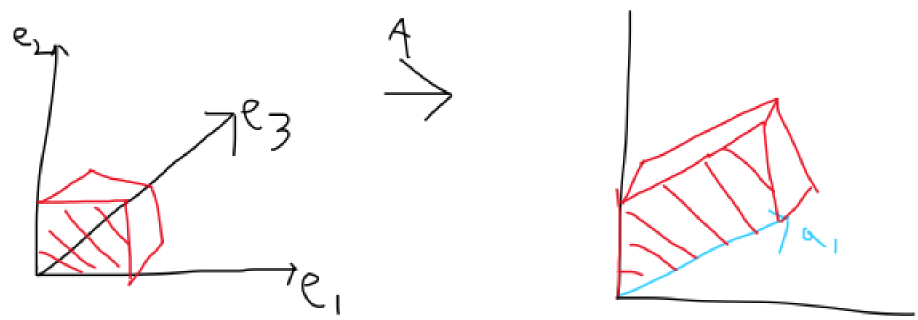
Leibniz Formula: Denote by S_n *the set of all permutations of 1,2,...n then* $det(A) = \Sigma_{\sigma \in Sn} * sign(\sigma) a_{1r(1)} * ... * a_{nr(n)}$
$\Sigma_{\sigma \in Sn}$ *is all permutations,* $sign(\sigma)$ *is the sign of a permutation,* $a_{1r(1)} * ... * a_{nr(n)}$ *is the position in the matrix.*

Special cases:
n = 1 det(a) = a
n = 2 det $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$
n=3 $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a * \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b * \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c * \begin{pmatrix} d & e \\ g & h \end{pmatrix}$

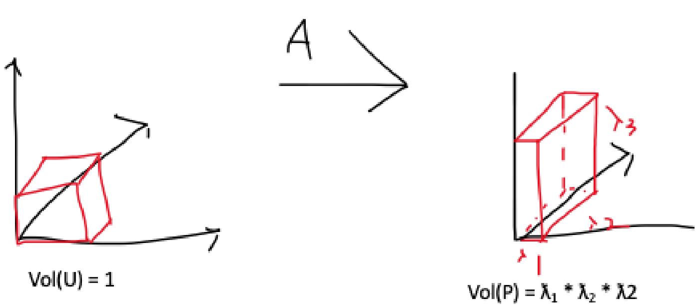
In general, there exists the formula of Laplace that expresses the determinant of a nxn matrix as a weighted linear combination of determinants of many (n-1) x (n-1) submatrices.
$det(a) =^n \Sigma_{j=1} * (-1)^{i+j} b_{ij} * det(B_{ij})$

1.5 Geometric Intuition

Consider a nxn matrix A with columns $(a_1, a_2, ..., a_n) = A$. *Consider the unit cube* $U = \{c_1e_1 + ... + c_ne_n | o <= c_i <= 1\}$



$U \longmapsto P := \{c_1a_1 + c_2a_2 + ... + c_na_n | o <= c_i <= 1\}$
Parallelotope



Then det(A) gives us the signed volume of the parallelotope P.
det(A) = product of eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A.
vol(U) changes by a factor of $\lambda_1 * \lambda_2 * \lambda_3$

1.6 Application to integrals

Proposition 10 *$\Omega \in \mathbb{R}$ open set, $\sigma : \omega \to \mathbb{R}^n$ differentiable, $f : \sigma(\omega) \to \mathbb{R}$. Then: $\int_{\sigma(\Omega)} f(y)\,dy$ =*

$$\int_\Omega f(\sigma(x)) * |det(\sigma'(x))|\,dx$$

Observation 11 *σ differentiable, that is we can locally (on a small ball B around x) approximate σ by a linear function.*

$$\sigma' = \begin{pmatrix} \frac{\delta\sigma_1}{\delta * x_1} & ... & \frac{\delta\sigma_1}{\delta * x_n} \\ ... & ... & ... \\ \frac{\delta\sigma_n}{\delta * x_1} & ... & \frac{\delta\sigma_n}{\delta * x_n} \end{pmatrix}$$

$vol(B) \approx vol(\sigma'(x)) * B)$
$\approx |det(\sigma'(x))| * vol(B)$
$f(y) * vol(B) \approx f(\sigma(x)) * det(\sigma'(x))| * vol(B)$

$$\int_{\sigma(\Omega)} f(y)\,dy$$

=

$$\int_\Omega f(\sigma(x)) * |det(\sigma'(x))|\,dx$$