1 Diagonalization

Definition 1 An operator $T \in \mathcal{L}(v)$ is diagonalizable if there exists a basis of $V$ such that the corresponding matrix is diagonal:

$$M(T) = \begin{pmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & & \lambda_n \end{pmatrix}$$

Nice property: Diagonal form is the best since we have the eigenvectors as the basis.

Proposition 2 Let $V$ be a finite-dimension vector space. $A \in \mathcal{L}(v)$. Then the following statements are equivalent:

$(P_1)$ $A$ is diagonalizable

$(P_2)$ The characteristic polynomial $P_A$ can be decomposed into linear factors AND The algebraic multiplicity of the roots of $P_A$ are equal to the geometric multiplicity

$(P_3)$ If $\lambda_1, \ldots, \lambda_k$ are the pairwise distinct eigenvalues of $A$, then

$$V = E(A, \lambda_1) \oplus E(A, \lambda_2) \cdots \oplus E(A, \lambda_k)$$

2 Triangular Matrices

Definition 3 A matrix is called upper triangular if it has the form

$$M(T) = \begin{pmatrix} \lambda_1 & * \\ & \ddots \\ 0 & & \lambda_n \end{pmatrix}$$

Proposition 4 $T \in \mathcal{L}(v), \Phi = \{v_1, v_2, \ldots, v_n\}$ a basis, then following are equivalent:

$(P_1)$ $M(T, D)$ is upper triangular

$(P_2)$ $Tv_j \in \text{span}\{v_1, v_2, \ldots, v_j\} \forall j = 1, 2, \ldots, n$
\[
Tv_1 = \begin{pmatrix}
\lambda_1 & a_{12} & a_{13} \\
0 & \lambda_2 & a_{23} \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 \\
0 \\
0
\end{pmatrix} = \lambda_1 \cdot v_1
\]

\[
Tv_2 = \begin{pmatrix}
\lambda_1 & a_{12} & a_{13} \\
0 & \lambda_2 & a_{23} \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
= \begin{pmatrix}
a_{12} \\
\lambda_2 \\
0
\end{pmatrix}
+ \lambda \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\in \text{span}(v_1, v_2)
\]

**Proposition 5**  
V complex, finite-dim VS, \(T \in \mathcal{L}(V)\). Then \(M(T)\) has an upper triangular form for some basis.

\[\rightarrow \text{If we are in the complex field, every matrix can be expressed as an upper triangular matrix.}\]

**Proposition 6**  
Suppose \(T \in \mathcal{L}(V), V\) any finite-dim VS, has an upper triangular form. Then the entries on the diagonal are precisely the eigenvalues of \(T\).

### 3 Metric Space

\[
\begin{align*}
\text{Metric} & \quad \rightarrow \quad \text{Normed} \\
\text{spaces} & \quad \rightarrow \quad \text{inner} \\
& \quad \rightarrow \quad \text{product} \\
& \quad \rightarrow \quad \text{Hilbert} \\
& \quad \rightarrow \quad \text{spaces}
\end{align*}
\]

\[K-N \quad \rightarrow \quad \text{Metric}\]

**Definition 7**  
Let \(x\) be a set. A function \(d: x \times x \rightarrow \mathbb{R}\) is called a metric if the following conditions hold. \(\forall u, v, w \in X:\)

\[(P_1) \ d(u, v) > 0 \text{ if } u \neq v \text{ and } d(u, u) = 0\]

\[(P_2) \ d(u, v) = d(v, u) \text{ (symmetry)}\]

\[(P_3) \ d(u, v) \leq d(u, w) + d(w, v)\]

**Example:** asymmetric measures

(i) friendship graph

(ii) \(KL(p\|q) \neq KL(q\|p)\)
**Notation:** Sequence: \((x_1, x_2 \ldots \rightarrow (x_n)_{n \in \mathbb{N}}

**Definition 8** Sequence: \((x_1, x_2 \ldots \rightarrow (x_n)_{n \in \mathbb{N}}

A sequence \((x_n)_{n \in \mathbb{N}}\) in a metric space \((x, d)\) is called a Cauchy Sequence if \(\forall \varepsilon > 0 \exists N \in \mathbb{N}, \forall n, m > N, d(x_n, x_m) < \varepsilon\)

**Definition 9** A sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x \in X\) if \(\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N, \ d(x_n, x) < \varepsilon\)

**Notation:** \(x_n \rightarrow x, \lim_{n \rightarrow \infty} x_n = x\)

Sequence \((x_n)_{n \in \mathbb{N}} = 1/n\) on \(x = (0,1)\)

Here \((x_n)_{n \in \mathbb{N}}\) is a Cauchy seq. But does not converge.

Sequence \((x_n)_{n \in \mathbb{N}} = 1/n\) on \(\tilde{x} = [0,1]\). Here \((x_n)_{n \in \mathbb{N}}\) is a cauchy sequence that converges on \(\tilde{x}\) to 0.

**Definition 10** A metric space is called complete if every Cauchy sequence converges.
**Notation:** \( B_\varepsilon(u) := \{ x \in X \mid d(x, u) \leq \varepsilon \} \to \varepsilon - \text{ball} \)

**Definition 11** A set \( A \subseteq X \) is called **closed** if all Cauchy sequences converge and have their limit point of \( A \).

**Definition 12** A set \( A \subseteq X \) is called **open** if:

\[
\forall a \in A \exists \varepsilon > 0 : B_\varepsilon(a) \subset A
\]

- **Set** \([0, 1]\) is closed.
- **Set** \((0, 1)\) is open.

\[
B = (a - \varepsilon, a + \varepsilon)
\]

- A set \( A \) can be neither open nor closed. e.g. \([0, 1)\).
Definition 13 A point \( a \in A \) is an interior point of \( A \) if \( \exists \varepsilon > 0 \), s.t. \( B_{\varepsilon}(a) \subset A \).

- e.g. \( A = [0, 1] \), then \( x \in (0, 1) \) are interior points.

Definition 14 The (topological) closure of a set \( A \) is defined as the set of points that can be approximated by Cauchy sequences in \( A \):

\[
\omega \in \bar{A} \iff \forall \varepsilon > 0 \exists z \in A : d(\omega, z) < \varepsilon
\]

Notation: \( \bar{A} \) is the closure of \( A \). \( A \cup dA \) (always closed!)

Definition 15 The (topological) interior of a set \( A \) is defined as the set of interior points of \( A \).

Notation: \( A^0 \)

Definition 16 The (topological) boundary of a set \( A \) is defined as the set \( \bar{A} \setminus A^0 \).

\[
\begin{align*}
\bar{x} &= [0, 1] \quad \text{sometimes} \\
x^0 &= (0, 1) = \{0\}
\end{align*}
\]

\( \Rightarrow \) boundary \( \partial x = \bar{x} \setminus x^0 = \{0, 1\} \)

Definition 17 A set \( A \) is dense in \( X \) if we can approximate every \( x \in X \) by a sequence in \( A \). Formally, \( \forall x \in X \quad \forall \varepsilon > 0, B_{\varepsilon}(x) \cap A \neq \emptyset \).

Example: \( \mathbb{Q} \subset \mathbb{R} \) is dense

Definition 18 A set \( A \subset X \) is bounded if there exists \( D > 0 \) such that \( \forall u, v \in A \ d(u, v) < D \).

4 Norms

Definition 19 Let \( V \) be a vector space. A norm on \( V \) is a function \( \| \cdot \| : V \to \mathbb{R} \) such that \( \forall x, y \in V, \lambda \in F \), the following conditions hold:

\[
(P_1) \| \lambda x \| = |\lambda| \| x \| \quad (\text{homogeneous})
\]

\[
(P_2) \| x + y \| \leq \| x \| + \| y \| \quad (\text{triangle inequality})
\]

\[
(P_3) \quad x = 0 \Rightarrow \| x \| = 0
\]

\[
(P_4) \quad \| x \| = 0 \Rightarrow x = 0
\]

\( \| \cdot \| \) is a semi-norm if \( P_1 \) - \( P_3 \) are satisfied.
**Intuition:** \( \text{norm}(x) = "\text{length of } x\)"

\[ = \text{distance}(x, 0) \]

**Examples:**

- **Euclidean norm on** \( \mathbb{R}^d \):
  \[ \|x\| = \left( \sum_{i=1}^{d} x_i^2 \right)^{1/2} \]

- **Manhattan distance**:
  \[ \|x\| = \left( \sum_{i=1}^{d} |x_i| \right) \]

5 **p-Norm**

Consider \( V = \mathbb{R}^d \). Define \( \| \cdot \|_p : \mathbb{R}^d \to \mathbb{R} \)

\[ \|x\|_p := \left( \sum_{i=1}^{d} |x_i|^p \right)^{1/p} \text{ for } 0 < p < \infty \]

- **\( \| \cdot \|_p \) is a norm if** \( p \geq 1 \)

- **Unit balls**: the unit ball of a norm is the set of points such that norm \( \leq 1 \):
  \[ B_p := \{ x \in \mathbb{R}^2 \mid \|x\|_p \leq 1 \} \]

**Examples:** \( \mathbb{R}^2 \)

\[ p = 2 \text{ (convex)} \]
\[ p = 1 \text{ (convex)} \]
\[ p = 0.5 \text{ (not convex)} \]

5-6
Definition 20 \( \|x\|_\infty := \max |x_i| \) (is a norm)

\( \|x\|_0 := \text{number of non-zero coordinates} = \sum_{i=1}^d \mathbb{1}\{x_i \neq 0\} \) \( \|x\|_0 \) is not a norm

\[
x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \|x\|_0 = 1; \lambda x = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \|\lambda x\|_0 = 1
\]

\[
\lambda = 5 \neq 5.1
\]