CSE 840: Computational Foundations of Artificial Intelligence September 18, 2023

Norms and Spaces of Functions

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1 Equivalence of Norms

Theorem 1 *All norms on $\mathbb{R}^n$ are (topologically) equivalent: If $||\cdot||_a$ and $||\cdot||_b$ are two norms on $\mathbb{R}^n$, then there exists constants $\alpha, \beta > 0$ such that:*

$$\forall x \in \mathbb{R}^n : \alpha||x||_a \leq ||x||_b \leq \beta||x||_a$$

Proof of Theorem 1: Without loss of generality (WLOG), we prove that if $||\cdot||$ is any norm on $\mathbb{R}^n$, then it is equivalent to $||\cdot||_\infty$ on $\mathbb{R}^n$.

First inequality: $\exists c_1 > 0 : \forall x \|x\| \leq c_1 \|x\|_\infty$

Let $x = \sum x_i e_i$ the representation of x in the standard basis of $\mathbb{R}^n$

$\|x\| = \| \sum_{i=1}^n x_i e_i \| \leq \sum_{i=1}^n \|x_i e_i\| = \sum_i |x_i| \|e_i\| \leq \sum_i \|x\|_\infty \|e_i\| = \|x\|_\infty \sum_i \|e_i\|$

$$\text{Second Inequality: } \exists c, > 0 \forall x : ||x||_\infty < c - ||x||_b$$

Let $S = \{x \in \mathbb{R}^n | ||x||_\infty = 1\}$ be the unit sphere w.r.t. $||\cdot||_\infty$. Consider $f : S \to \mathbb{R}, f(x) > ||x||_a$. The mapping f is continuous w.r.t. $||\cdot||_\infty$; this follows from the fact that:

$$||f(x) - f(y)|| = |\|x - y\|| \leq \|x - y\| \leq C_1 \|x - y\|_\infty$$

(Lipschitz continuity).
S is closed and bounded, so S is compact (from analysis). Any continuous mapping on a compact set takes its min and max. Define $m = \min f(x) | x \in S$. Then $||x||_a = m||x||_b$. Since $x \in S$, $||x||_b = 1$. Thus, $m \leq ||x||_a \leq M$ where M is the max of f on S.

2 Convex Sets are Unit balls of norms

Definition 2 *Consider a real vector space, V. A set S is called convex if $\forall$ b $\in$ [0,1] and $\forall$ x,y $\in$ S,*

$$b \cdot x + (1 - b) \cdot y \in S$$

Definition 3 *A set $C \subset v$ is called symmetric if $z \in C \implies$ -x $\in$ C.*

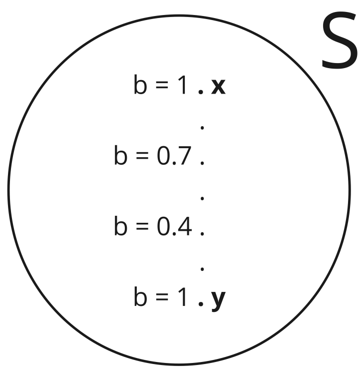

Figure 1: A convex set holds all the points between x and y

Figure 2: A set cannot be convex if values lie outside the set

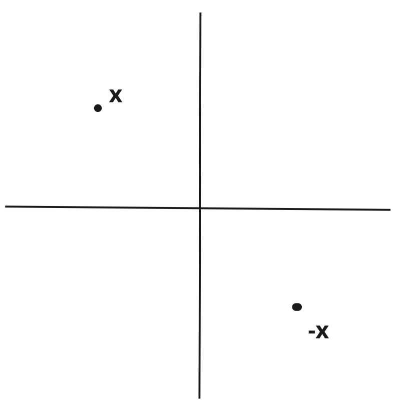

Figure 3: Example of symmetric set

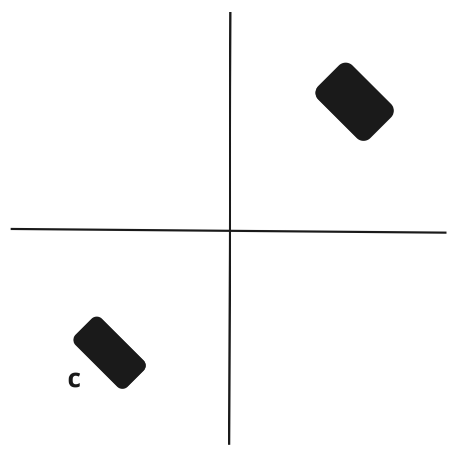

Figure 4: Both are a part of set, c, and symmetric

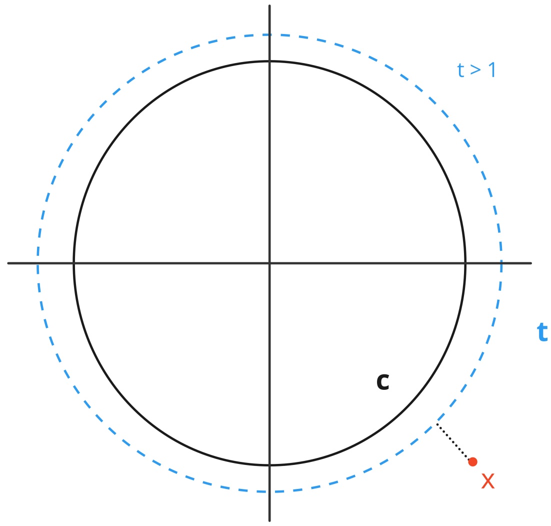

Figure 5: The smallest factor, t, needed to multiply c to reach x

Theorem 4 *1) Let $C \subset \mathbb{R}^n$ be closed, convex, symmetric, and has non-empty interior. Define $P(x) := \inf\{\lambda > 0 \,|\, x \in \lambda * C\}$. Then P is a semi-norm. If C is bounded, then P is a norm, and its unit ball coincides with C, i.e., $C = \{x \in \mathbb{R}^n \,|\, P(x) \leq 1\}$.*

2) For any norm $||\cdot||$ on $\mathbb{R}^n$, the set $C = \{x \in \mathbb{R}^n \,|\, ||x|| \leq 1\}$ is bounded, symmetric, closed, convex, and has non-empty interior.

Proof of Theorem 4: p(x) is well defined

Want to prove: given x $\in \mathbb{R}^d$, the set

$$\{t > 0 | x \in t \cdot c\} \neq \varnothing$$

We are going to prove:

$$\exists\ \varepsilon > 0 \text{ such that}$$

$$B_\varepsilon(0) = \{e \in \mathbb{R}^d | \|e\|_2 < \varepsilon\}$$

Intuition:

•By assumption, C has at least one interior point.

$$v \in C^0 => \exists \varepsilon \text{ such that}$$

$$B_\varepsilon(v) \in C => v + B_\varepsilon(0) = \{v + e | e \in B_\varepsilon(0)\}$$

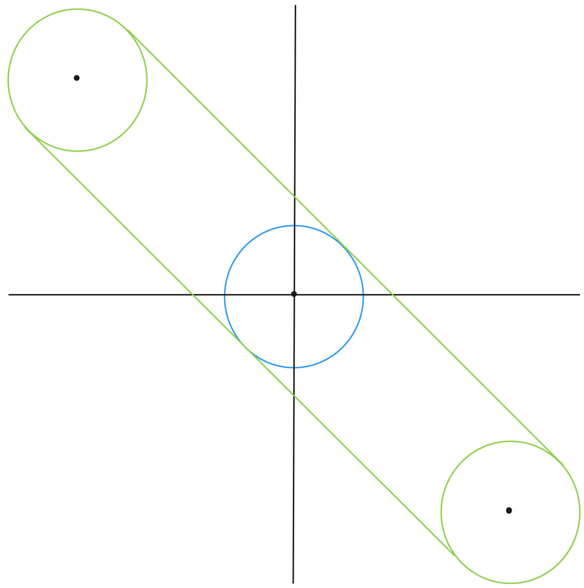

Figure 6: Unit Norm at the origin from a ball containing an interior point

By symmetry,

$$v + e \in C \Rightarrow -(v + e) \in C$$

By convexity,

$$\frac{1}{2}(v + e) + \frac{1}{2}(-(v + e)) = e \in C$$

So, $B_\epsilon(0) \subset C$, so the set $\{t > 0 \mid x \in t \cdot C\}$ is non-empty.
The infimum of $\inf\{t > 0 \mid x \in t \cdot C\}$ exists because

$$\{t > 0 \mid x \in t \cdot C\} \subset \mathbb{R}$$

has 0 as its lower bound.

(P1)

$$P(0) = 0$$

- have seen: $0 \in c$
- $\forall t > 0 : 0 \in 0 \cdot C$
- $\inf\{t \mid 0 \in t \cdot C\} = 0$

$$\Rightarrow P(0) = 0$$

(P2)

$$P(\alpha x) = |\alpha| P(x)$$

For all $\alpha > 0$, we have

$$p(\alpha \cdot x) = \inf\{t > 0 \mid \alpha \cdot x \in t \cdot C\}$$

$S := \frac{t}{\alpha}$,

$$= \inf\{\alpha \cdot S > 0 \mid x \in S \cdot C\}$$

$$= \alpha \cdot \inf\{S > 0 \mid x \in S \cdot C\}$$

$$= \alpha P(x)$$

$$=> P(\alpha x) = |\alpha| P(x)$$

By symmetry we also get

$$P(-x) = P(x)$$

Combining the two statements to say

$$P(\alpha x) = |\alpha| P(x)$$

(homogeneity)

(P3) Triangle - Inequality.

Consider $x, y \in \mathbb{R}^d, s, t > 0$ such that:

$$\frac{x}{s} \in C, \frac{y}{t} \in C.$$

Observe:

$$\frac{s}{s+t} + \frac{t}{s+t} = 1$$

Then, by convexity,

$$\frac{s}{s+t}.\frac{x}{s} + \frac{t}{s+t}.\frac{y}{t} \in C$$

Because, $\frac{s}{s+t}$ and $\frac{t}{s+t}$ are two scalars that sum to 1.

And that $\frac{x}{s}$ and $\frac{y}{t} \in C$

$$=> \frac{x+y}{s+t} \in C$$

$$=> \frac{x+y}{u_0} \in S$$

$$=> P(x+y) = \inf\{u > 0 \mid x+y \in u.C\} \leq u_0 \leq s+t$$

$$= P(x) + P(y)$$

s was chosen such that $x \in s.C$

t was chosen such that $y \in t.C$

Consider a sequence $(s_i)_{i \in \mathbb{N}}$ such that $x \in s_i \cdot c$ and $s_i \to p(x)$.

Similarly, $(t_i)_{i \in \mathbb{N}}$ such that $y \in t_i \cdot c$ and $t_i \to p(y)$.

$$\forall i : P(x+y) \leq s_i + t_i$$

Knowing $t_i \to p(y)$ and $s_i \to p(x)$

Since it satisfies all i, then it has to be valid for the limit points.

$$\Rightarrow P(x+y) \leq P(x) + P(y)$$

Property 4. $\quad P(x) = 0 \Rightarrow x = 0.$

$P(x) = 0 \Longleftrightarrow inf\{t > 0 | x \in t \cdot c\} = 0$

$\Rightarrow$ There exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \to 0$ and $x \in t_k \cdot c \quad \forall k.$

Now assume that $x \neq 0$. Then the sequence $\left(\frac{x}{t_k}\right)_{k\in\mathbb{N}}$ is unbounded.

$\Rightarrow$ contradiction since by definition we know that c is bounded.

2.1 Normed Function Spaces

2.1.1 Space of continuous functions

Definition 5 *Let T be a metric space,*

$$e^b(T) := \{f : T \to \mathbb{R} \,|\, f \text{ is continuous and bounded}\}$$

Here, bounded means: $\longrightarrow (\exists c \in \mathbb{R} : \forall t \in T : |f(t)| < c)$

As norm on $e^b(T)$ we choose:

$$\|f\|_\infty := \sup_{t\in T} |f(t)|$$

The norm exists since we are in the space of bounded functions, bounded from above.

Then the space $e^b(T)$ with norm $\|\cdot\|_\infty$ is called a Banach Space.

A more general version of the Banach Space: If $(x, d_{\|\cdot\|_\infty})$ is a complete metric, then the normed space $(x, \|\cdot\|_\infty)$ is called a Banach Space.

Proof outline:

1. Needs to check vector space axioms.
2. Norm axioms.
3. **Completeness**: follows from the fact that $\|f\|_\infty$ induces uniform convergence. (Here, a sequence of functions will converge in this norm that we are using if the limit point is an element of the space)

2.1.2 Space of differentiable functions

Definition 6 *Let* $[a, b] \subset \mathbb{R}, \quad e'([a, b]) = \{f : [a, b] \to \mathbb{R} | f$ *is continuously differentiable* $\}$

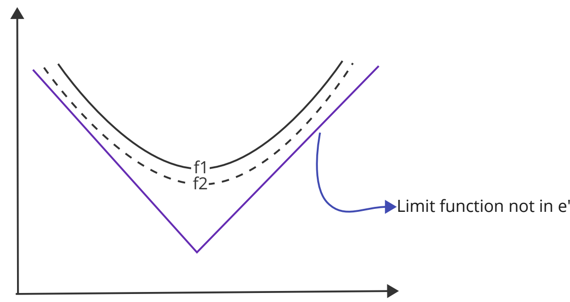

Figure 7: Caption

<u>Which norm?</u>

Consider $\| \cdot \|_\infty$ With this norm, e' is not complete. As shown in Fig 7, we can create a sequence of functions f_1 and f_2, etc. We can make these functions to be as close as we want to the limit function. We can never get to the limit function, which is not continuous.

Is there a better norm? The answer is Yes. Many norms exist. Let us consider a few examples.

Consider $\|f\| := \sup_{t\in[a,b]} max\{|f(t)|, |f'(t)|\}$

Consider $\|f\| := \|f\|_\infty + \|f'\|_\infty$

$e'([a,b])$ with any of these two norms is a Banach Space.