CSE 840: Computational Foundations of Artificial Intelligence	February 19, 2025
Sequences and Convergence	
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1 Introduction

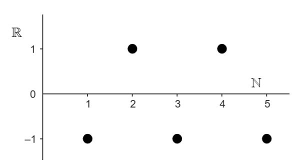
Calculus is fundamental for machine learning. Derivatives are often used for optimization problems whereas integrals are used for finding the expected behavior. All of which are based on finding the limit:

$$\lim_{b \to a} \left(\frac{f(b) - f(a)}{b - a} \right)$$

2 Sequences

2.1 Examples

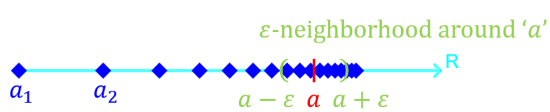
a. $(a_n)_{n \in \mathbb{N}} = ((-1)^n)_{n \in \mathbb{N}} = (-1, 1, -1, 1, ...)$



b. $(a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} = (1, \frac{1}{2}, frac13, ...)$ $\quad \lim_{n \to \infty} a_n = 0$

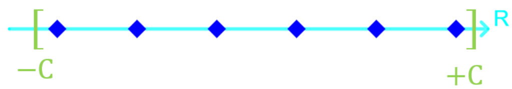
c. $(a_n)_{n \in \mathbb{N}} = (2^n)_{n \in \mathbb{N}} = (2, 4, 8, 16, ...)$

Definition 1 *A sequence $(a_n)_{n \in \mathbb{N}}$ is called convergent to $a \in \mathbb{R}$ if $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N :$ $|a_n - a| < \varepsilon$*



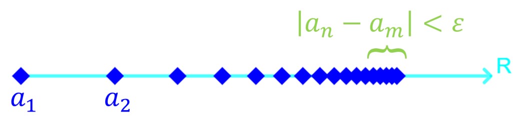
If there is no such $a \in \mathbb{R}$, then sequence diverges.

Definition 2 *A sequence $(a_n)_{n \in \mathbb{N}}$ is called bounded if $\exists C \in \mathbb{R} \quad \forall n \in \mathbb{N} : |a_n| \leq C$ otherwise, the sequence is unbounded.*



Fact 3 *$(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ bounded*
$(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ There is only one limit $\lim_{n\to\infty} a_n = a$

Definition 4 *If $\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq \mathbb{N} : |a_n - a_m| < \varepsilon$. then $(a_n)_{n\in\mathbb{N}}$ is called a Cauchy Sequence.*



Fact 5 *For sequence of real numbers: Cauchy sequence $\Leftrightarrow$ convergent sequence*

Proposition 6 *If $(a_n)_{n\in\mathbb{N}}$ is montonically decreasing $(a_{n+1} \leq a_n \quad \forall n)$ and bounded from below (the set $\{a_n\}_{n\in\mathbb{N}}$ has a lower bound), then $(a_n)_{n\in\mathbb{N}}$ is convergent. Example subsequence: $(a_n)_{n\in\mathbb{N}} = (-1)^n$ Subsequence: $(a_n)_{n\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}} = (1, 1, ..., 1) \quad \to 1$ subsequence: $(a_n)_{n\in\mathbb{N}} = (a_{2k+1})_{k\in\mathbb{N}} = (-1, -1, ...) \quad \to -1$*

Definition 7 *$a \in \mathbb{R}$ is called an accumulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subsequence $(a_{n_k})_{K\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a$. (cluster point accumulation point, limit point, partial limit*

Theorem 8 *Bolzano-Weierstrass theorem. $(a_n)_{n\in\mathbb{N}}$ bounded $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value. (has a convergent subsequence).*

Observation 9

- *a sequence can have many accumulation points or no accumulation point.*
- *Even if the sequence has just one accumulation point, it is not necessarily a Cauchy sequence.*
- *If $(a_n)_{n\in\mathbb{N}}$ converges to a, then a is the only accumulation point, and the sequence is a Cauchy sequence.*

Example: $(a_n)_{n\in\mathbb{N}} = \frac{1}{n}$ on $(0, 1]$ $(a_n)_{n\in\mathbb{N}}$ is Cauchy, but does not converge on $(0, 1]$. It does converge to 0 on $[0, 1]$

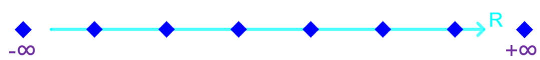
2.2 Max, Sup, Min, Inf

Assume we are on $\mathbb{R}$ (or more generally, on a space that has on a total ordering). Let $U \subset \mathbb{R}$ be a subset.

- $x \in \mathbb{R}$ is called a maximum element of U if $x \in U$ and $\forall u \in U : U \leq x$. For example, 1 is the max [0, 1] while (0,1) has no max.
- x is Called an upper bound of U if $\forall u \in U : u \leq x$. For example, 5 is an upper bound of both (0,1) and (0,1].
- x is called a supremum of U if it is the smallest upper bound. For example, 1 is the sup of (0,1).

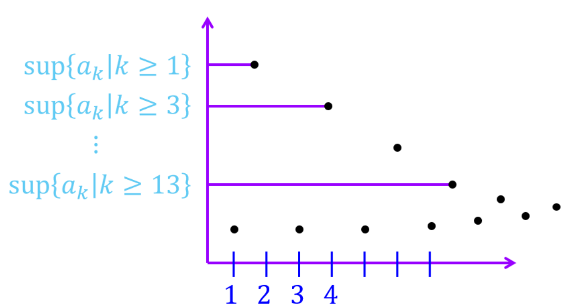
Analogously define minimum, lower bound and infimum.

A given sequence $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values where as $+\infty, -\infty$ are called improper accumulation points.



Definition 10 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. An element $a \in \mathbb{R} \cup \{\infty, +\infty\}$ is called:*

- *limit superior of $(a_n)_{n\in\mathbb{N}}$ if a is the largest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$ write $a = \limsup_{n\to\infty} a_n$*
- *limit inferior of $(a_n)_{n\in\mathbb{N}}$ if a is the Smallest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$ write $a = \liminf_{n\to\infty} a_n$*



When looking at all a function such as the one shown by the points above, we can evaluate the supremum of the limit by funding the supremum of the sequence as the starting index increases to infinity.

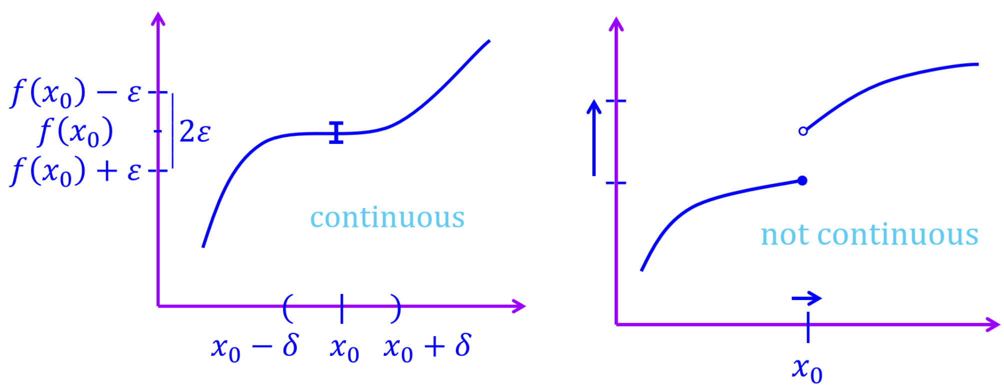
Fact 11

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$

3 Continuity

Definition 12 *A function $f : X \to Y$ between two metric spaces (x, d), (Y, d) is called* <u>*continuous at $x_0 \in X$*</u> *if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X: \quad d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$



Definition 13 *Alternative Def: $f : X \to Y$ is called continuous at x_0 if for every sequence $(a_n)_{n \in \mathbb{N}} \subset X$ we have: $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$*

Definition 14 *A function $f : X \to Y$ is called* <u>*continuous*</u> *if it is continuous for every $x_0 \in X$:*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X: \quad d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

Definition 15 *A function $f : X \to Y$ is called* <u>*Lipschitz continuous*</u> *with Lipschitz constant L if*

$$\forall x, y \in X : d(f(x), f(y)) \leq L \cdot d(x, y)$$

Intuition: bounded derivative

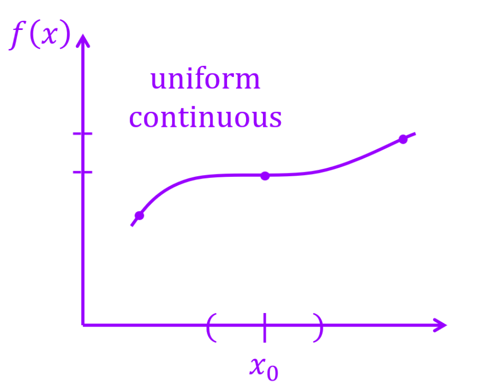
Definition 16 *A function $f : X \to Y$ is called* <u>*uniformly continuous*</u> *if*

$$\forall x, y \in X : d(f(x), f(y)) \leq L \cdot d(x, y)$$

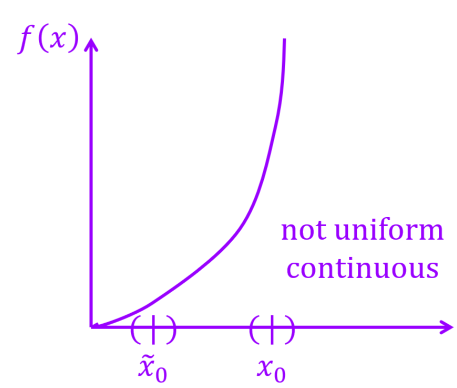
4 Important Theorems for continuous Functions

Theorem 17 <u>*Intermediate value theorem*</u>*: If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains all values between $f(a)$ & $f(b)$:*

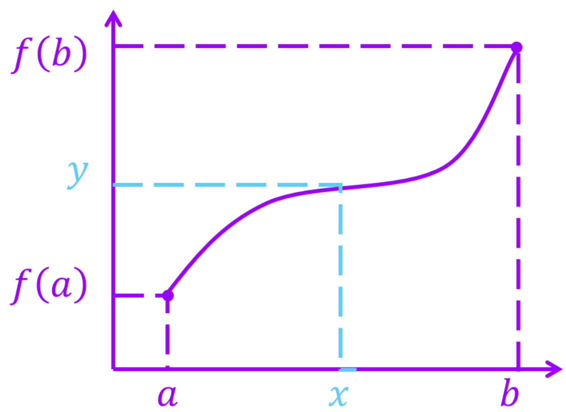
$$\forall y \in [f(a), f(b)] \exists x \in [a, b] : f(x) = y$$



Given ε, can choose δ that works for all x_0
Intuition: bounded derivative



Given ε, cannot choose δ to be the same for all x_0
Intuition: unbounded derivative



Application 18 *If you want to find x with $f(x) = 0$ find a with $f(a) < 0$, b with $f(b) > 0$ then there must exist $x \in [a, b]$ with $f(x) = 0$*

Definition 19 *<u>Invertible Functions</u>: $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ continuous, strictly monotone ($a < b \Rightarrow f(a) < f(b)$) Then f is invertible and the inverse is continuous as well.*

- Invertible follows from monotonicity
- continuity of the inverse follows directly from continuity of f.

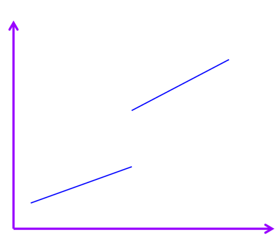
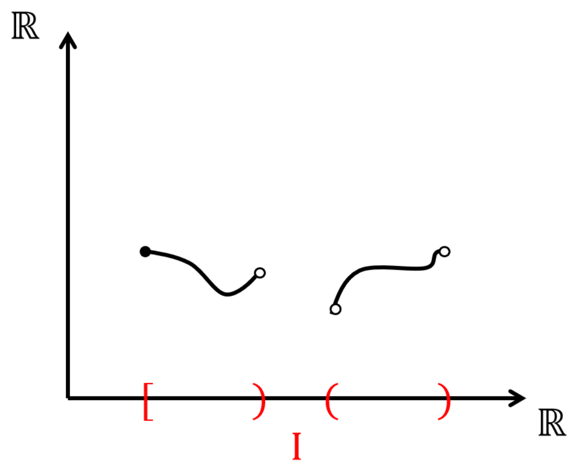


Figure 1: An example of an invertible function that is not continuous.

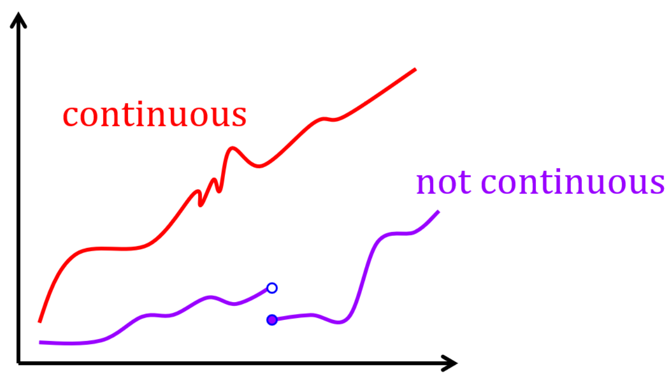
A function f between two metric spaces (x, d), (y, d) is continuous if and only if pre-images of open sets are open:

$$B \subset Y \text{ open in } y \Rightarrow f^{-1}(B) := \{x \in x \mid f(x) \in B\} \text{ open in } x$$

Function: $f : I \to \mathbb{R} \quad (I \in \mathbb{R})$



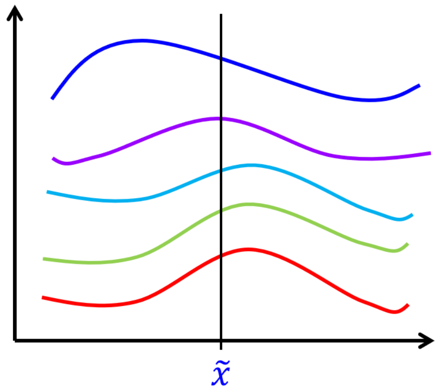
Continuous function: $f : \mathbb{R} \to \mathbb{R}$



Idea: small changes on x-axis → small changes on y-axis

5 Sequences of functions

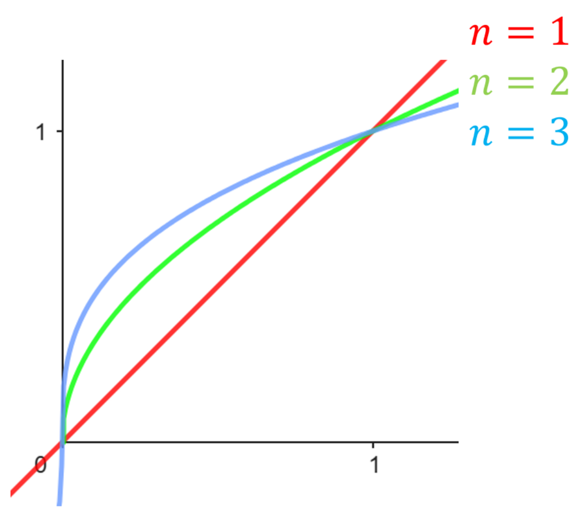
Consider the sequence: $(f_1, f_2, f_3, f_4, ...)$ with members $f_1 : I \to \mathbb{R}$, $f_2 : I \to \mathbb{R}$, ... as show below.



For any fixed $\tilde{x} \in I$, we can get an ordinary sequence of real-numbers. $(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), f_4(\tilde{x}), ...)$

Definition 20 *Consider functions:* $f_n : I \to \mathbb{R}, I \subseteq \mathbb{R}$ *We say that the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges pointwise to* $f : I \to \mathbb{R}$ *if* $\forall x \in I : f_n(x) \to f(x)$
$y_n := f_n(x),\ y = f(x)$
$y_n \to y$

Example: $f_n, f : [0,1] \to \mathbb{R}, \quad f_n(x) = x^{1/n}$



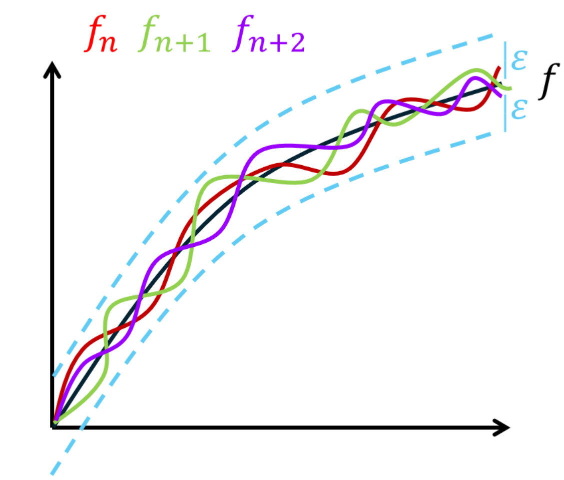
$$f(x) = \begin{cases} 0 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$f_n \to f$ pointwise, all f_n continuous, this does not imply that f is continuous

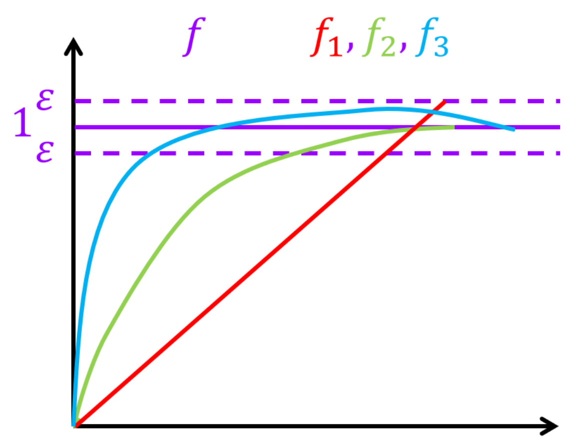
Definition 21 $(f_n)_{n\in\mathbb{N}}$ *converges to* f *uniformly if* $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \exists x \in I : |f_n(x) - f(x)| < \varepsilon$

Intuition:

Uniform convergence: given ε, there exists N such that all $f_n \quad n > N$ are contained within ε-tube



Close to zero, there will always be points x close to zero such that the $f_n(x)$ are not yet in E-tube. f fr, furfs Not uniformly convergence.



Definition 22 *Alternative definition: $f_n \to f$ uniformly iff $\|f_n - f\|_\infty \to 0$.*

Theorem 23 *(uniform convergence preserves continuity) $f_n, f : I \to \mathbb{R}, \quad I \subset \mathbb{R}$, all f_n are continuous, $f_n \to f$ uniformly. Then f is continuous.*

6 Sample Questions

a. Show that $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges to 0 in $\mathbb{R}$.

To prove this, we must choose N such that $|a_n - a| < \varepsilon$. As such, we find that $|\frac{1}{n} - 0| < \varepsilon$, which we rearrange to say that $n > \frac{1}{\varepsilon}$. Therefore, if we pick N such that $N > \frac{1}{\varepsilon}$, then we always get $|a_n - a| < \varepsilon$.

b. Show that $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$.

There are two possible ways to prove this. The first is to pick $\varepsilon = 1/2$ and follow the steps in the previous proof. The second option is to show that $(1, 1, ...)$ and $(-1, -1, ...)$ are both convergent subsequences of a_n. Since a_n therefore has two different accumulation points, it cannot converge.

c. Is the sequence $C_n = \frac{2n^2+5n-1}{-5n^2+n+1}$ convergent?.

We face the initial problem that both the numerator and denominator tend to infinity, leaving us with an indeterminate form. To circumvent this, We can multiply the numerator and denominator by $\frac{1}{n^2}$:

$$C_n = \frac{2n^2 + 5n - 1}{-5n^2 + n + 1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{2 + \frac{5}{n} - \frac{1}{n^2}}{-5 + \frac{1}{n} + \frac{1}{n^2}}$$

In this form, as n tends to infinity, $1/n$ tends to zero and we are left C_n converging to $-2/5$.