CSE 840: Computational Foundations of Artificial Intelligence	Feburary 24, 2025
Derivatives and Riemann Integral	
Instructor: Vishnu Boddeti	*Scribe: Cherilyn Wang*

1 Derivatives (1-dimensional case)

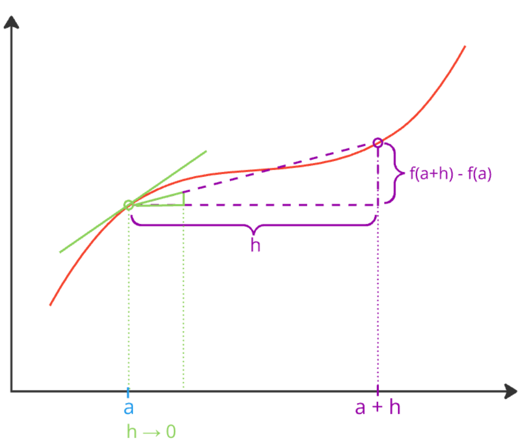
Definition 1 *Consider $U \subset \mathbb{R}$ an interval, $f : U \to \mathbb{R}$. The function is called differentiable at $a \in U$ if:*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

We often write

$$f' = \frac{df}{dx}$$



$$a_n : a + h_n, h_n \to 0, h_n > 0$$

"Derivative is the slope of a function at point a"
"Slope of linear approximation of a function at point a"

$$f(x) = f(a) + (x - a)\underline{b} \text{ (b is the slope and the derivative at a)}$$

The function is called differentiable if it is differentiable for all $a \in$ U. It is continuously differentiable if it is differentiable and the function $f' : U \to \mathbb{R}, a \mapsto f'(a)$ is continuous.

Higher derivatives: We can repeat the process of taking derivatives:

$$f' = \frac{df}{dx}, f'' = \frac{df'}{dx}$$

Notation: $f^{(n)}$ denotes the n-th derivative (if exists).

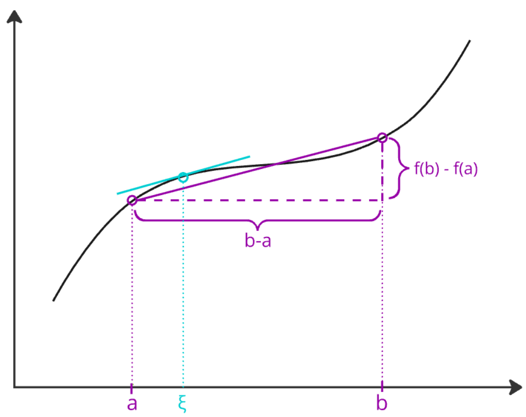
1.1 Important Theorems

Theorem 2 *Differentiable $\Rightarrow$ continuous, but continuous $\not\Rightarrow$ differentiable*

Let f be differentiable at a, then there exists a constant C_a such that on a small ball around a, we have $|f(x) - f(a)| \leqslant C_a |x - a|$. In particular, f is continuous at a (This is also the case globally: if the function is differentiable everywhere, it is also continuous everywhere).

Theorem 3 *Intermediate value theorem for derivatives*

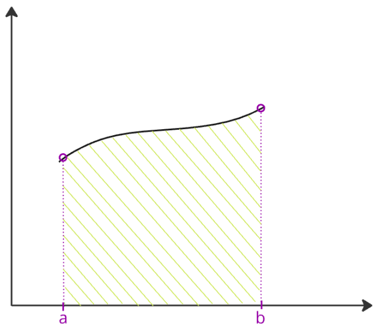
$f \in \mathcal{C}'([a,b])$(i.e. functions on [a,b] that are once continuously differentiable), then there exists $\xi \in [a,b]$ such that $\frac{f(b)-f(a)}{b-a} = f'(\xi)$



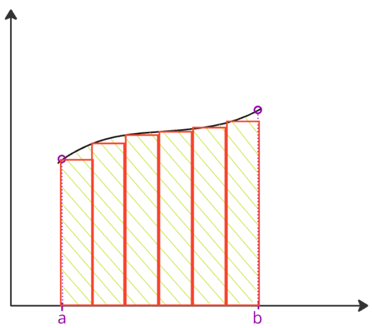
Theorem 4 *exchanging limits and derivatives $f_n : [a,b] \to \mathbb{R}, f_n \in \mathcal{C}'([a,b])$ if the limit $f(x) := \lim\limits_{n\to\infty} f_n(x)$ exists $\forall x \in [a,b]$ and the derivatives f' converge uniformly, then f is continuously differentiable and we have:*

$$\begin{aligned} f'(x) &= (\lim_{n\to\infty} f_n)'(x) \to \textit{first take limit of } f_n\textit{, we obtain } f\textit{, then compute derivative} \\ &= (\lim_{n\to\infty} (f_n'))(x) \to \textit{first compute } f_n'\textit{, then take the limit} \end{aligned} \tag{1}$$

2 Riemann Integral

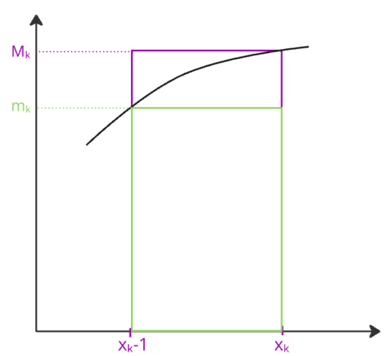


Consider a function $f : [a, b] \to \mathbb{R}$, assume that f is bounded $(\exists l, u \in \mathbb{R}, \forall x \in [a, b] : l \leqslant f(x) \leqslant u)$



Consider $x_0, x_1, ..., x_n$ with $a = x_0 < x_1 < x_2 ... < x_n = b$. These points introduce a partition of $[a, b]$ into n intervals.

$$I_k := [x_k - 1, x_k]$$



Define $m_k := inf(f(I_k))$, $M_k := sup(f(I_k))$ (exists since f is bounded).

Define the lower sum

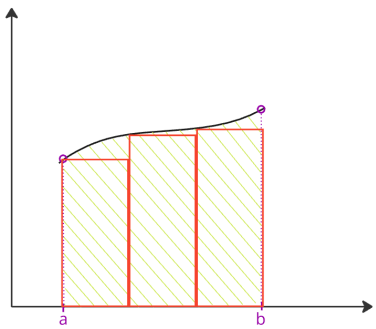
$$s(f, \{x_0, x_1, ...x_n\}) = \sum_{k=1}^{n} |I_k| \cdot m_k \text{ (length of } I_k = x_k - x_{k-1})$$

Define the upper sum

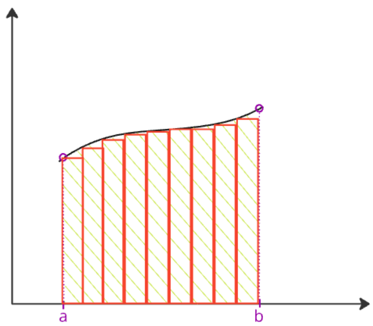
$$s(f,\{x_0,x_1,...x_n\}) = \sum_{k=1}^{n} |I_k| \cdot M_k$$

Now define

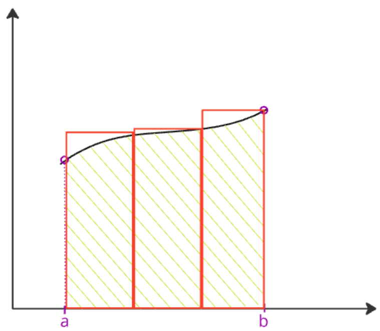
$$\mathcal{J}_* := \underset{partitions}{sup} \quad (s(f, partition))$$
$$\mathcal{J}^* := \underset{partitions}{inf} \quad (s(f, partition))$$



coarse partition from below



finer partition from below



partition from above

We call f Riemann-integrable if $\mathcal{J}_* = \mathcal{J}^*$. Then we denote

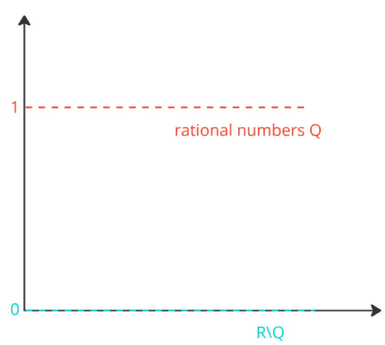
$$\mathcal{J}_* = \mathcal{J}^* := \int_a^b f(t)dt$$

Theorem 5
- *$f : [a,b] \to \mathbb{R}$, monotone $\Rightarrow$ integrable* (i.e. $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$)
- *$f : [a,b] \to \mathbb{R}$, continuous $\Rightarrow$ integrable (true even if f is continuous everywhere except at finitely many points)*

Shortcomings:

- Many functions are not integrable

$$\text{Dirichlet Function} : f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & everywhere \end{cases}$$



For any interval $I_k = [x_k, x_{k+1}], M_k = 1, m_k = 0$

Then $\mathcal{J}_* < \mathcal{J}^*$ ($\mathcal{J}_*$:$|b-a| \cdot 0$, $\mathcal{J}^*$:$|b-a| \cdot 1$)

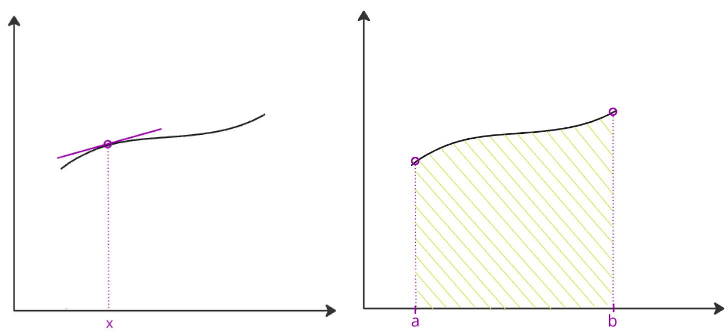
- One cannot prove theorems about exchanging "integral" with "limit":

$$\lim_{n\to\infty} \int f_n dt \stackrel{?}{=} \int \lim_{n\to\infty} f_k dt$$

- Hard to extend to "other space" (e.g. spaces with no notion of ordering, higher dimensional)

Later in the course, we will see that Lebesque Integration, which is a more modern and more useful form of integration, can overcome some of the shortcomings of Riemann Integration.

3 Fundamantal Theorem of Calculus



Derivatives are intuitively slopes of the function at a particular point x, while integration computes the area under the curve. However, it is not very obvious how and why these two are related to each other. The fundamental theorem of calculus tries to relate these two.

Theorem 6 *$f : [a, b] \to \mathbb{R}$ is (Riemann)-integrable and continuous at $\xi \in [a, b]$. Let $c \in [a, b]$. Then the function*

$$F(x) := \int_c^x f(t)dt$$

is differentiable at ξ and $F'(\xi) = f(\xi)$. If $f \in \mathcal{C}([a, b])$, then $F \in \mathcal{C}'([a, b])$ and $F'(x) = f(x)$ for all $x \in [a, b]$

Proof: This is the theorem's proof. □

Theorem 7 *$F : [a, b] \to \mathbb{R}$ is continuously differentiable, then*

$$\int_a^b F'(t)dt = F(b) - f(a)$$